VECTOR VALUED SPHERICAL FUNCTIONS AND MACDONALD-KOORNWINDER POLYNOMIALS

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Abstract. We interpret the five parameter family of Macdonald-Koornwinder polynomials as vector valued spherical functions on quantum Grassmannians.

Introduction

The representation theoretic construction of (quantum) conformal blocks in certain conformal field theories is closely related to harmonic analysis on (quantum) symmetric spaces of group type, see, e.g., [5] and [4]. A striking consequence is the interpretation of A-type Macdonald polynomials as vector valued spherical functions for the quantum analogue of the symmetric pair $(\mathrm{U}(n)\times\mathrm{U}(n), \mathrm{diag}(\mathrm{U}(n)))$ of group type on the one hand (see [3]), and as quantum conformal blocks on the other hand (see [4]). With these interpretations many properties of A-type Macdonald polynomials, such as the Macdonald-Ruijsenaars difference equations, quantum Khnizhnik-Zamolodchikov equations, dualities and orthogonality relations, obtain their natural representation theoretic and conformal field theoretic interpretations. In this paper we consider the harmonic analytic part of these constructions for the quantum analogues of the symmetric pair $(U,K)=(\mathrm{U}(2n),\mathrm{U}(n)\times\mathrm{U}(n))$. This leads to the interpretation of the five parameter family of Macdonald-Koornwinder polynomials as vector valued spherical functions.

In [22] the classical analogue of our main result was established. It yields the interpretation of BC-type Heckman-Opdam polynomials as the restriction to the maximal torus of regular vector valued functions

$$f: U \to \mathbb{C}\mathrm{det}^{-\kappa_1}\otimes S^{n\kappa}(\mathbb{C}^n)\mathrm{det}^{\kappa_1-\kappa}, \qquad \kappa_1\in\mathbb{Z},\, \kappa\in\mathbb{Z}_{\geq 0}$$

(with $S^{n\kappa}(\mathbb{C}^n)$ the homogeneous polynomials of degree $n\kappa$) which transform under the left (respectively right) regular K-action on U according to the natural K-action on the image space (respectively the K-character $\mathrm{det}^{-\kappa_2}\otimes\mathrm{det}^{\kappa_2}$ for some $\kappa_2\in\mathbb{Z}$).

In this paper we define a continuous one-parameter family of quantum analogues of the symmetric pair (U,K), following closely Letzter's [11]–[15] approach of constructing quantum symmetric pairs as coideal subalgebras. One may as well view this family of quantum symmetric pairs as a continuous one-parameter family of quantum analogues of the complex Grassmannian U/K. We proceed by defining the analogue of the K-representation $\mathbb{C}\mathrm{det}^{-\kappa_1}\otimes S^{n\kappa}(\mathbb{C}^n)\mathrm{det}^{\kappa_1-\kappa}$ for the associated coideal subalgebras. This allows us to define vector valued spherical functions for the one-parameter family of quantum symmetric pairs in essentially the same manner as in the classical case [22], with the exception that we have now the additional freedom to choose different coideal subalgebras

for the transformation behaviour under the left and right regular action, respectively. We relate the resulting vector valued spherical functions, which now depend on two continuous and three discrete parameters, to the five parameter family of Macdonald-Koornwinder polynomials. The Macdonald-Koornwinder polynomials are Koornwinder's [10] extension of the BC-type Macdonald polynomials that contain all Macdonald polynomials of classical type as special cases.

The interpretation of the five parameter family of Macdonald-Koornwinder polynomials as vector valued spherical functions contains several known results as special cases. It entails the interpretation of a two parameter subfamily of Macdonald-Koornwinder polynomials as zonal spherical functions on quantum Grassmannians, which was established by Noumi, Sugitani and Dijkhuizen [19] (see also [2]). This special case plays in fact an essential role in establishing our general result. In rank one we reobtain the interpretation of the four parameter family of Askey-Wilson polynomials as matrix coefficients of quantum sl(2) representations, established before by Koornwinder [9] (zonal case), and by Noumi and Mimachi [20] and Koelink [8] (general case).

The content of the paper is as follows. In §1 we give the main definitions and formulate the main result. In §2 we define the notion of expectation value for the quantum symmetric pairs under investigation and establish branching rules using deformation theory. In §3 we prove the zonal case by translating the main results of [19] to our setup. In §4 we generalize the Chevalley restriction theorem to the setup of vector valued spherical functions. Its description involves the vector valued spherical function of the smallest degree, which we call the ground state (in the zonal case the ground state is the unit). The restriction of the ground state to the quantum torus is computed explicitly in §5. In §6 we establish the quantum Schur orthogonality relations for the vector valued spherical functions and, combined with the results of previous sections, we establish the explicit interpretation of the Macdonald-Koornwinder polynomials as vector valued spherical functions.

The construction of expectation values in §2 and the dynamical quantum group interpretation of the rank one results in [25] hint at a natural interpretation of the Macdonald-Koornwinder polynomials as quantum conformal blocks. A more detailed study in this direction is subject of future research.

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1. Formulation of the main result

In this section we give definitions of the Macdonald-Koornwinder polynomials, the quantum symmetric pairs and the associated class of vector valued spherical functions, and we

formulate the main result of the paper. We add insightful proofs of some of the intermediate results, but we postpone the more technical parts to later sections. We fix a positive integer $n \geq 1$ and a deformation parameter $0 < q < 1$ throughout the paper.

1.1. **Macdonald-Koornwinder polynomials.** Koornwinder [10] extended the definition of the Macdonald polynomials [17] associated to the non-reduced, irreducible root system BC_n to a family of orthogonal polynomials depending, besides on the deformation parameter q, on five additional coupling parameters. This family of polynomials, known nowadays as Macdonald-Koornwinder polynomials, reduces for $n = 1$ to the celebrated, four parameter family of Askey-Wilson polynomials. In this subsection we recall their definition.

Denote Λ_n for the lattice $\mathbb{Z}^n$ and let $\Lambda_n^+ \subset \Lambda_n$ be the set of partitions of length $\leq n$. The dominance partial order on Λ_n is defined by

$$\lambda \leq \mu \qquad \Leftrightarrow \qquad \sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \mu_i, \quad j = 1, \ldots, n.$$

The Weyl group $W = S_n \ltimes \{\pm 1\}^n$ of BC_n, where S_n is the symmetric group in n letters, acts on Λ_n by permutations and sign changes. Each W-orbit in Λ_n intersect Λ_n^+ exactly once.

Let $\mathbb{C}[u^{\pm 1}] = \mathbb{C}[u_1^{\pm 1}, \ldots, u_n^{\pm 1}]$ be the algebra of Laurent polynomials in n independent variables u_k $(1 \leq k \leq n)$, or, equivalently, the algebra of regular functions on the complex n-torus $T^{\mathbb{C}} = (\mathbb{C}^*)^n$. A basis of $\mathbb{C}[u^{\pm 1}]$ is given by the monomials $u^\lambda = u_1^{\lambda_1} u_2^{\lambda_2} \cdots u_n^{\lambda_n}$ $(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Lambda_n)$.

The Weyl group W acts on $\mathbb{C}[u^{\pm 1}]$ by permutations and inversions of the u_k. Let $\mathbb{C}[u^{\pm 1}]^W \subset \mathbb{C}[u^{\pm 1}]$ be the subalgebra of W-invariant Laurent polynomials. The orbit sums $m_\lambda = \sum_{\mu \in W\lambda} u^\mu$ $(\lambda \in \Lambda_n^+)$ form a linear basis of $\mathbb{C}[u^{\pm 1}]^W$.

The Macdonald-Koornwinder polynomials form an orthogonal basis of $\mathbb{C}[u^{\pm 1}]^W$ with respect to a particular scalar product on $\mathbb{C}[u^{\pm 1}]^W$. The scalar product, which we define now first, depends on five additional coupling parameters a, b, c, d and t. It is defined in terms of an absolutely continuous measure with respect to the normalized Haar measure on the natural compact real form $T = \mathbb{T}^n$ of $T^{\mathbb{C}} = (\mathbb{C}^*)^n$, where $\mathbb{T}$ is the unit circle in the complex plane. The corresponding weight function Δ is most conveniently expressed in terms of the q-shifted factorial,

$$(a; q)_k = \prod_{i=0}^{k-1} (1 - aq^i), \qquad \forall k \in \mathbb{Z}_+ \cup \{\infty\}$$

by $\Delta(u) = \Delta^+(u)\Delta^+(u^{-1})$ with $\Delta^+(u) = \Delta^+(u; a, b, c, d; q, t)$ defined by

$$\Delta^+(u) = \prod_{i=1}^{n} \frac{(u_i^2; q)_\infty}{(au_i, bu_i, cu_i, du_i; q)_\infty} \prod_{1 \leq i < j \leq n} \frac{(u_i/u_j, u_iu_j; q)_\infty}{(tu_i/u_j, tu_iu_j; q)_\infty}.$$

Here $(a_1,\ldots,a_s;q)_k=\prod_{j=1}^s(a_j;q)_k$ is a short-hand notation for products of q-shifted factorials. If a,b,c,d,t are real and

$$|a|,|b|,|c|,|d|<1, \quad 0<t<1, \tag{1.1}$$

then $\Delta(u)$ is a positive continuous weight function on T. Under these assumptions, we can define the Macdonald-Koornwinder polynomials as follows, see [10].

Theorem 1.1. *There exist unique W-invariant Laurent polynomials*

$$P_\lambda(u)=P_\lambda(u;a,b,c,d;q,t)\in\mathbb{C}[u^{\pm 1}]^W, \qquad \lambda\in\Lambda_n^+$$

satisfying the two conditions

$$P_\lambda(u)=m_\lambda+\sum_{\mu\in\Lambda_n^+:\mu<\lambda}c_{\lambda\mu}m_\mu, \qquad \textit{some}\ \ c_{\lambda\mu}\in\mathbb{C},$$
$$\int_T P_\lambda(u)\overline{P_\mu(u)}\Delta(u)\frac{du}{u}=0, \qquad \textit{if}\ \ \lambda\neq\mu,$$

where $\frac{du}{u}=\frac{du_1}{u_1}\frac{du_2}{u_2}\cdots\frac{du_n}{u_n}$ is the Haar measure on T. We call $P_\lambda(u)$ the monic Macdonald-Koornwinder polynomial of degree $\lambda\in\Lambda_n^+$.

The theorem does not follow by a straightforward Gram-Schmidt type procedure since the dominance ordering is not a total ordening. The key tool in proving the theorem is an explicit self-adjoint difference operator which maps a symmetric monomial m_λ to a linear combination of symmetric monomials m_μ involving only degrees $\mu\leq\lambda$ (see [10]). In our set-up, the self-adjoint operator arises as the radial part of the quadratic Casimir element for the corresponding quantum symmetric pair (see [19, Thm. 3.3]).

Remark 1.2. Theorem 1.1 is also valid for generic values of the parameters a,b,c,d outside the region (1.1) after deforming the compact torus T in the definition of the orthogonality relations in a suitable way. In general one looses positivity of the weight function, but for suitable values of the parameters (still violating the condition that the modulus of all the four parameters a,b,c and d is less than one), one can reobtain a positive measure by shifting the deformed compact torus to T while picking up residues, see [24].

1.2. **The quantized universal enveloping algebra.** We introduce here the notations for the quantized universal enveloping algebra of $\mathfrak{gl}(m)$. For further details and standard facts, we refer to [21] and [18].

In notations below we surpress the dependence on $m\in\mathbb{Z}_{>0}$ as much as possible. For most of our applications, m will be either $2n$ (in which case we write $\mathfrak{g}$ for the Lie algebra $\mathfrak{gl}(2n)$), or m will be n (in which case we stick to the notation $\mathfrak{gl}(n)$). Let $\delta_{i,j}$ be the usual Kronecker delta function ($=1$ if $i=j$, and $=0$ otherwise). The quantized universal enveloping algebra $U_q(\mathfrak{gl}(m))$ is the unital algebra over $\mathbb{C}$ generated by $K_i^{\pm 1}$ ($i=1,\ldots,m$),

x_j, y_j ($j = 1, \ldots, m-1$), subject to the relations

$$
\begin{aligned}
&K_iK_j = K_jK_j, && K_iK_i^{-1} = 1 = K_i^{-1}K_i,\\
&K_ix_jK_i^{-1} = q^{\delta_{i,j}-\delta_{i,j+1}}x_j, && K_iy_jK_i^{-1} = q^{-\delta_{i,j}+\delta_{i,j+1}}y_j,\\
&x_iy_j - y_jx_i = \frac{K_iK_{i+1}^{-1} - K_i^{-1}K_{i+1}}{q-q^{-1}}\,\delta_{i,j}, &&\\
&x_ix_j = x_jx_i \quad \text{and} \quad y_iy_j = y_jy_i && \text{when } |i-j| \geq 2,\\
&x_i^2x_j - (q+q^{-1})x_ix_jx_i + x_jx_i^2 = 0, && \text{when } |i-j| = 1,\\
&y_i^2y_j - (q+q^{-1})y_iy_jy_i + y_jy_i^2 = 0, && \text{when } |i-j| = 1.
\end{aligned}
$$

The quantized universal enveloping algebra $U_q(\mathfrak{gl}(m))$ is a Hopf $*$-algebra, with comultiplication

$$
\begin{aligned}
\Delta(x_j) &= x_j \otimes 1 + K_jK_{j+1}^{-1} \otimes x_j,\\
\Delta(y_j) &= y_j \otimes K_j^{-1}K_{j+1} + 1 \otimes y_j,\\
\Delta(K_i^{\pm 1}) &= K_i^{\pm 1} \otimes K_i^{\pm 1},
\end{aligned}
$$

counit

$$\varepsilon(x_j) = \varepsilon(y_j) = 0, \qquad \varepsilon(K_i^{\pm 1}) = 1,$$

antipode

$$S(x_j) = -K_j^{-1}K_{j+1}x_j, \qquad S(y_j) = -y_jK_jK_{j+1}^{-1}, \qquad S(K_i^{\pm 1}) = K_i^{\mp 1} \tag{1.2}$$

and $*$-structure

$$x_j^* = q^{-1}y_jK_jK_{j+1}^{-1}, \qquad y_j^* = qK_j^{-1}K_{j+1}x_j, \qquad (K_i^{\pm 1})^* = K_i^{\pm 1}.$$

This $*$-structure corresponds classically to choosing the skew-hermitean matrices $\mathfrak{u}(m)$ as (compact) real form of $\mathfrak{gl}(m)$. We use the (modified) Sweedler notation $\Delta(X) = \sum X_1 \otimes X_2$ ($X \in U_q(\mathfrak{g})$) for the comultiplication.

1.3. **The quantum analogues of the symmetric pair** $(\mathfrak{gl}(2n), \mathfrak{gl}(n) \times \mathfrak{gl}(n))$**.** The realization of a two parameter subfamily of the Macdonald-Koornwinder polynomials as zonal spherical functions on quantizations of the complex Grassmannian by Noumi, Dijkhuizen and Sugitani [19], see also [2], play a crucial role in the present paper. A one-parameter family of quantum Grassmannians is defined in [19] in terms of invariance properties with respect to an one-parameter family of two-sided coideals, which are analogues of the Lie subalgebra $\mathfrak{k} = \mathfrak{gl}(n) \times \mathfrak{gl}(n) \subset \mathfrak{g} = \mathfrak{gl}(2n)$. Although there is a natural way to define the analogue of the trivial representation of $\mathfrak{k}$ to the coideal setup, this is no longer the case for other representations. To get a grip on the non-trivial representations, we replace the two-sided coideals by coideal subalgebras of $U_q(\mathfrak{g})$, an idea which was suggested by Noumi [18] and developed in full generality in a series of papers [11], [12], [13], [14], [15], [16] by Letzter. For the quantum symmetric pair $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{gl}(2n), \mathfrak{gl}(n) \times \mathfrak{gl}(n))$ the coideal algebras have a particularly nice presentation in terms of generators and relations, closely

resembling the relations of the Drinfeld-Jimbo quantized universal enveloping algebras (see [12] and [15]).

One of the more technical parts of this paper is the translation of the results of Noumi, Dijkhuizen and Sugitani [19] in the language of right coideal algebras. Unfortunately, in doing so it turns out to be more convenient to work with a slightly modified version of Letzter's right coideal algebra. In this subsection we define the modified right coideal algebra for the symmetric pair $(\mathfrak{g},\mathfrak{k})$, and we present some of its important properties.

Let $\{\epsilon_i\}_{i=1}^{2n}$ be the standard orthonormal basis of the Euclidean space $\big(\mathbb{R}^{2n},(\cdot,\cdot)\big)$ and denote

$$\alpha_j=\epsilon_j-\epsilon_{j+1},\qquad j=1,\ldots,2n-1.$$

Definition 1.3. *Let $\mathcal{A}$ be the unital, associative algebra over $\mathbb{C}$ with generators $\gamma_i^{\pm1}$ ($i=1,\ldots,2n$) and β_j ($j=1,\ldots,2n-1$) satisfying the relations*

$$\gamma_i\gamma_j=\gamma_j\gamma_i,\qquad \gamma_i\gamma_i^{-1}=1=\gamma_i^{-1}\gamma_i,\qquad \gamma_i=\gamma_{2n+1-i}\tag{1.3}$$

for $i,j=1,\ldots,2n$;

$$\gamma_i\beta_j=q^{-(\epsilon_i+\epsilon_{2n+1-i},\alpha_j)}\beta_j\gamma_i\tag{1.4}$$

for $i=1,\ldots,2n$ and $j=1,\ldots,2n-1$;

$$\beta_i\beta_j-\beta_j\beta_i=q\left(\frac{\gamma_i^{-2}-\gamma_{i+1}^{-2}}{q-q^{-1}}\right)\delta_{i,2n-j}\tag{1.5}$$

for $i,j=1,\ldots,2n-1$ with $|i-j|\geq 2$;

$$q^{-1}\beta_i^2\beta_{i+1}-(q+q^{-1})\beta_i\beta_{i+1}\beta_i+q\beta_{i+1}\beta_i^2=q\beta_{n+1}\gamma_n^{-2}\delta_{i,n}\tag{1.6}$$

for $i=1,\ldots,2n-2$ and

$$q\beta_i^2\beta_{i-1}-(q+q^{-1})\beta_i\beta_{i-1}\beta_i+q^{-1}\beta_{i-1}\beta_i^2=q^{-1}\beta_{n-1}\gamma_n^{-2}\delta_{i,n}\tag{1.7}$$

for $i=2,\ldots,2n-1$.

Observe that $\mathcal{A}$ becomes a $*$-algebra with $*$-structure defined by

$$\gamma_i^*=\gamma_i,\qquad\qquad \beta_j^*=\beta_{2n-j}$$

for $i=1,\ldots,2n$ and $j=1,\ldots,2n-1$.

Proposition 1.4. *Let $\sigma\in\mathbb{R}$. The assignment*

$$\begin{aligned}\pi_\sigma(\gamma_i^{\pm1})&=K_i^{\pm1}K_{2n+1-i}^{\pm1},\\ \pi_\sigma(\beta_j)&=y_jK_{j+1}^{-1}K_{2n-j}^{-1}+K_j^{-1}x_{2n-j}K_{2n-j}^{-1},\\ \pi_\sigma(\beta_n)&=y_nK_{n+1}^{-1}K_n^{-1}+K_n^{-1}x_nK_n^{-1}+\left(\frac{q^{-\sigma}-q^{\sigma}}{q-q^{-1}}\right)K_n^{-2}\end{aligned}\tag{1.8}$$

for $i\in\{1,\ldots,2n\}$ and $j\in\{1,\ldots,2n-1\}\setminus\{n\}$, uniquely extends to an injective $*$-algebra homomorphism $\pi_\sigma:\mathcal{A}\to U_q(\mathfrak{g})$.

Proof. This is essentially [15, Thm. 7.1] for the symmetric pair of type AIII (Case 2), see [15, §7]. More specifically, following the proof of [11, Lem. 2.2] one shows that the assignment (1.8) uniquely extends to a unital $*$-algebra homomorphism $\pi_\sigma : \mathcal{A} \to U_q(\mathfrak{g})$ for $\sigma \in \mathbb{R}$. The injectivity of π_σ follows by a straightforward modification of the proof of [11, Prop. 2.3]. □

Definition 1.5. *Let $\sigma \in \mathbb{R}$. We call $(U_q(\mathfrak{g}), \mathcal{A}_\sigma)$ with $\mathcal{A}_\sigma = \pi_\sigma(\mathcal{A})$, a quantum analogue of the symmetric pair $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{gl}(2n), \mathfrak{gl}(n) \times \mathfrak{gl}(n))$. We denote*

$$C_i = \pi_\sigma(\gamma_i), \qquad B_j = \pi_\sigma(\beta_j), \qquad B_n^\sigma = \pi_\sigma(\beta_n) \tag{1.9}$$

($i \in \{1, \ldots, 2n\}$ and $j \in \{1, \ldots, 2n-1\} \setminus \{n\}$) for the image of the generators of $\mathcal{A}$ under the embedding π_σ.

We fix $\sigma \in \mathbb{R}$ once and for all, unless specified differently.

The terminology in Definition 1.5 can formally be justified as follows, cf., e.g., [11, §3]. Taking the classical limit $q \to 1$ of the (modified) generators of $\mathcal{A}_\sigma$ gives

$$\begin{aligned} &\frac{C_i - C_i^{-1}}{q - q^{-1}} \to E_{i,i} + E_{2n+1-i,2n+1-i}, \\ &B_j \to E_{j+1,j} + E_{2n-j,2n+1-j}, \\ &B_n^\sigma \to E_{n+1,n} + E_{n,n+1} - \sigma 1 \end{aligned} \tag{1.10}$$

for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, 2n-1\} \setminus \{n\}$, where $E_{i,j} \in \mathfrak{g} \subset U(\mathfrak{g})$ is the matrix unit with a one in row i and column j, and zeroes elsewhere. The unital subalgebra $\widetilde{\mathcal{A}}$ of $U(\mathfrak{g})$ generated by the classical limits (1.10) of the generators of $\mathcal{A}_\sigma$ is independent of σ and isomorphic to the subalgebra $U(\mathfrak{k}) \subset U(\mathfrak{g})$. The isomorphism is induced by the inner automorphism $\mathrm{Ad}(g)$ of $\mathfrak{g}$, with $g \in \mathrm{GL}(2n; \mathbb{C})$ given explicitly by

$$g = \frac{1}{\sqrt{2}} \sum_{i=1}^{n} \big(E_{i,i} + E_{i,2n+1-i} + E_{2n+1-i,i} - E_{2n+1-i,2n+1-i} \big). \tag{1.11}$$

In fact, the associated inner automorphism of $U(\mathfrak{g})$ preserves all classical generators (1.10) of $\widetilde{\mathcal{A}}$ besides $E_{n+1,n} + E_{n,n+1} - \sigma 1$, which is mapped to $E_{n,n} - E_{n+1,n+1} - \sigma 1$.

The classical isomorphism $\widetilde{\mathcal{A}} \simeq U(\mathfrak{k})$ via the inner automorphism $\mathrm{Ad}(g)$ has the following weak analogue on the quantum group level. Let $U_q(\mathfrak{k})$ be the Hopf $*$-subalgebra of $U_q(\mathfrak{g})$ generated by $K_i^{\pm 1}$ ($i \in \{1, \ldots, 2n\}$) and x_j, y_j ($j \in \{1, \ldots, 2n-1\} \setminus \{n\}$). Denote

$$\widehat{B}_n^\sigma = \frac{q^{-\sigma} K_{n+1}^{-2} - q^\sigma K_n^{-2}}{q - q^{-1}} \in U_q(\mathfrak{k}), \tag{1.12}$$

which has $E_{n,n} - E_{n+1,n+1} - \sigma 1$ as classical limit.

Proposition 1.6. *Let $I \subseteq U_q(\mathfrak{k})$ be the two-sided $*$-ideal generated by $x_{n-1} y_{n+1}$. The assignment*

$$\phi_\sigma(\gamma_i) = C_i + I, \qquad \phi_\sigma(\beta_j) = B_j + I, \qquad \phi_\sigma(\beta_n) = \widehat{B}_n^\sigma + I$$

for $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,2n-1\}\setminus\{n\}$ uniquely extends to a $*$-algebra homomorphism $\phi_\sigma : \mathcal{A} \to U_q(\mathfrak{k})/I$.

Proof. In view of Proposition 1.4, ϕ_σ extends to an algebra homomorphism $\phi_\sigma : \mathcal{A} \to U_q(\mathfrak{k})/I$ if ϕ_σ respects the defining relations of $\mathcal{A}$ involving β_n. This is a straighforward, but tedious computation. The $*$-ideal I plays only a role for the relations (1.6) for $i=n-1$ and relation (1.7) for $i=n+1$ (which in turn are each others $*$-images). In $U_q(\mathfrak{k})$ we have the relation

$$q^{-1}B_{n-1}^2\widehat{B}_n^\sigma-(q+q^{-1})B_{n-1}\widehat{B}_n^\sigma B_{n-1}+q\widehat{B}_n^\sigma B_{n-1}^2=q^{-\sigma-1}(q^{-2}-q^2)x_{n+1}y_{n-1}K_{n-1}^{-1}K_n^{-1}K_{n+1}^{-4},$$

which, modulo I, reduces to the ϕ_σ-image of relation (1.6) for $i=n-1$. The case $i=n+1$ is checked similarly. It is clear that the algebra homomorphism ϕ_σ respects the $*$-structure. □

Remark 1.7. Denote $U_q(\mathfrak{gl}(2))^{(n)}\subset U_q(\mathfrak{g})$ for the copy of the unital Hopf-$*$-algebra $U_q(\mathfrak{gl}(2))$ in $U_q(\mathfrak{g})$, generated by $x_n, y_n, K_n^{\pm1}, K_{n+1}^{\pm1}$. Note that both $B_n^\sigma\in U_q(\mathfrak{gl}(2))^{(n)}$ and $\widehat{B}_n^\sigma\in U_q(\mathfrak{gl}(2))^{(n)}$, with $\widehat{B}_n^\sigma$ given by (1.12). By [23], there exists an invertible element x_σ in a suitable completion of $U_q(\mathfrak{gl}(2))^{(n)}$ such that $x_\sigma B_n^\sigma x_\sigma^{-1}=\widehat{B}_n^\sigma$. Explicitly, x_σ can be given as formal infinite series by

$$x_\sigma=\sum_{l,m=0}^{\infty}\frac{q^{-(l+m)\sigma}(-1)^m}{\big(-q^{2-2\sigma};q^2\big)_l}\frac{q^{l^2+2lm-l-m}}{\big(q^2;q^2\big)_l\big(q^2;q^2\big)_m}(1-q^2)^{l+m}x_n^l\big(y_nK_nK_{n+1}^{-1}\big)^m. \tag{1.13}$$

The element x_σ is essentially Babelon's [1] vertex-IRF transformation, which leads to an interpretation of the parameter σ as a dynamical parameter in the sense of dynamical quantum groups, see [25] for a detailed discussion and further references. Note furthermore that conjugating by x_σ fixes all other generators B_j and $C_i^{\pm1}$ of $\mathcal{A}_\sigma$ besides B_{n-1} and B_{n+1}.

Let $\mathcal{E}\subset\mathcal{A}_\sigma\subset U_q(\mathfrak{g})$ be the unital subalgebra generated by the σ-independent generators $C_i^{\pm1}$ ($i\in\{1,\ldots,2n\}$) and B_j ($j\in\{1,\ldots,2n-1\}\setminus\{n\}$) of $\mathcal{A}_\sigma$. We end this subsection by proving that $\mathcal{E}$ is essentially the subalgebra $U_q(\mathfrak{gl}(n))$, diagonally embedded in $U_q(\mathfrak{g})$.

Throughout this paper we identify

$$U_q(\mathfrak{gl}(n))\otimes U_q(\mathfrak{gl}(n))\simeq U_q(\mathfrak{k})\subset U_q(\mathfrak{g})$$

as $*$-Hopf algebras, by identifying $x_j\otimes 1, y_j\otimes 1$ and $K_i^{\pm1}\otimes 1$ (respectively $1\otimes x_j, 1\otimes y_j$ and $1\otimes K_i^{\pm1}$) with x_j, y_j and $K_i^{\pm1}$ (respectively x_{n+j}, y_{n+j} and $K_{n+i}^{\pm1}$) for $j=1,\ldots,n-1$ and $i=1,\ldots,n$. Let $\psi: U_q(\mathfrak{gl}(n))\to U_q(\mathfrak{gl}(n))$ be the $*$-algebra isomorphism defined by

$$\psi(x_j)=y_{n-j},\qquad \psi(y_j)=x_{n-j},\qquad \psi(K_i^{\pm1})=K_{n+1-i}^{\pm1} \tag{1.14}$$

for $j=1,\ldots,n-1$ and $i=1,\ldots,n$. Let Δ^{op} be the opposite comultiplication of $U_q(\mathfrak{gl}(n))$.

Lemma 1.8. *For $j=1,\ldots,n-1$ and $i=1,\ldots,n$ we have*

$$(id\otimes\psi)\Delta^{op}(y_jK_{j+1}^{-1})=B_j,$$
$$(id\otimes\psi)\Delta^{op}(x_jK_j^{-1})=B_{2n-j},$$
$$(id\otimes\psi)\Delta^{op}(K_i^{\pm1})=C_i^{\pm1}.$$

Furthermore, $(id \otimes \psi)\Delta^{op}$ defines a $*$-algebra isomorphism

$$(id \otimes \psi)\Delta^{op} : U_q(\mathfrak{gl}(n)) \xrightarrow{\sim} \mathcal{E} \subseteq \mathcal{A}_\sigma \cap U_q(\mathfrak{k}) \subset U_q(\mathfrak{g}).$$

Proof. The proof is straightforward. □

Other crucial properties of the quantum symmetric pairs $(U_q(\mathfrak{g}), \mathcal{A}_\sigma)$, such as the coideal property of $\mathcal{A}_\sigma \subset U_q(\mathfrak{g})$, are discussed in later sections.

1.4. **Representations.** We discuss those aspects of the representation theory associated to the quantum symmetric pairs $(U_q(\mathfrak{g}), \mathcal{A}_\sigma)$ which are relevant for the purposes of this paper. We start by recalling some standard facts on the finite dimensional representation theory of the quantized universal enveloping algebra $U_q(\mathfrak{gl}(m))$. If no confusion can arise, we surpress in the following definitions the dependence on m as much as possible.

Let $P_m = \bigoplus_{i=1}^m \mathbb{Z}\varepsilon_i \simeq \mathbb{Z}^{\times m}$ be the rational character lattice of $\mathrm{GL}(m;\mathbb{C})$ and

$$P_m^+ = \{(\lambda_1, \ldots, \lambda_m) \in P_m \,|\, \lambda_1 \geq \cdots \geq \lambda_m\}$$

the associated cone of dominant weights.

We only consider finite dimensional left $U_q(\mathfrak{gl}(m))$-modules M with weights in P_m, i.e. we assume that M has weight decomposition

$$M = \bigoplus_{\mu \in P_m} M[\mu],$$
$$M[\mu] = \{v \in M \,|\, K_i v = q^{\mu_i} v \ \ i = 1, \ldots, m\}.$$

Any such finite dimensional $U_q(\mathfrak{g})$-module M is $U_q(\mathfrak{g})$-semisimple. The irreducible ones are the irreducible finite dimensional highest weight representations L_λ of $U_q(\mathfrak{gl}(m))$ with highest weight $\lambda \in P_m^+$.

The irreducible module L_λ ($\lambda \in P_m^+$) has a one-dimensional weight space $L_\lambda[\lambda]$. A vector v_λ spanning $L_\lambda[\lambda]$ is called a highest weight vector of L_λ. A highest weight vector v_λ is a cyclic vector of L_λ satisfying $x_j v_\lambda = 0$ for $j = 1, \ldots, m-1$. In particular,

$$L_\lambda = \bigoplus_{\mu \in P_m : \mu \preceq \lambda} L_\lambda[\mu],$$

with $\preceq$ the A-type dominance order on P_m: $\lambda \preceq \mu$ for $\lambda, \mu \in P_m$ if

$$\sum_{i=1}^j \lambda_i \leq \sum_{i=1}^j \mu_i \quad (j \in \{1, \ldots, m-1\}), \qquad \sum_{i=1}^m \lambda_i = \sum_{i=1}^m \mu_i.$$

There exists a scalar product $\langle \cdot, \cdot \rangle_\lambda$ on L_λ, unique up to constant multiples, such that

$$\langle Xv, w \rangle_\lambda = \langle v, X^* w \rangle_\lambda, \qquad \forall\, v, w \in L_\lambda, \ \ \forall\, X \in U_q(\mathfrak{gl}(m)).$$

Consequently, any finite dimensional $U_q(\mathfrak{gl}(m))$-module M is $*$-unitarizable.

Definition 1.9. *We write M^σ for a left $U_q(\mathfrak{g})$-module M, viewed as $\mathcal{A}_\sigma$-module by restriction of the action to the subalgebra $\mathcal{A}_\sigma \subset U_q(\mathfrak{g})$.*

Since $\mathcal{A}_\sigma \subset U_q(\mathfrak{g})$ is $*$-invariant, we have the following result, cf., e.g., [13, Thm. 3.3].

Lemma 1.10. *If M is a finite dimensional $U_q(\mathfrak{g})$-module, then M^σ is $\mathcal{A}_\sigma$-semisimple.*

An abstract representation theory for $\mathcal{A}_\sigma$ was developed by Letzter [13, §5-7] in the general context of quantum symmetric pairs. For our purposes it is more convenient to have concrete realizations of certain special $\mathcal{A}_\sigma$-representations. These special representations are constructed using the following corollary of Proposition 1.6.

Corollary 1.11. *Let V be a finite dimensional Hilbert space and let $\rho : U_q(\mathfrak{k}) \to End_{\mathbb{C}}(V)$ be a $*$-representation such that $I \subseteq Ker(\rho)$. Denote by $\overline{\rho} : U_q(\mathfrak{k})/I \to End_{\mathbb{C}}(V)$ the associated $*$-representation of $U_q(\mathfrak{k})/I$. Then*

$$\rho_\sigma = \overline{\rho} \circ \phi_\sigma \circ \pi_\sigma^{-1} : \mathcal{A}_\sigma \to End(V_\sigma),$$

with $V_\sigma = V$ as finite dimensional Hilbert space, defines a $*$-representation of $\mathcal{A}_\sigma$ such that $\rho_\sigma|_{\mathcal{E}} = \rho|_{\mathcal{E}}$.

Remark 1.12. With the conventions of Corollary 1.11 we can define a $*$-representation $\rho_{\sigma,\tau} = \overline{\rho} \circ \phi_\tau \circ \pi_\sigma^{-1}$ of $\mathcal{A}_\sigma$ for arbitrary $\sigma, \tau \in \mathbb{R}$. The $*$-representation $\rho_{\sigma,\tau}$ thus satisfies

$$\rho_{\sigma,\tau}(B_n^\sigma) = \rho(\widehat{B}_n^\tau).$$

The special role of $\rho_\sigma = \rho_{\sigma,\sigma}$ follows from the fact that B_n^σ and $\widehat{B}_n^\sigma$ are conjugate in $U_q(\mathfrak{g})$ (cf. Remark 1.7) and from the fact that the formal classical limit of $\rho_\sigma = \rho_{\sigma,\sigma}$ is equivalent to the formal classical limit of ρ via the isomorphism $\widetilde{\mathcal{A}} \simeq U_q(\mathfrak{k})$ constructed in §1.3.

Let $\kappa_1 \in \mathbb{Z}$ and $\kappa \in \mathbb{Z}_{\geq 0}$. Denote $V(\kappa, \kappa_1)$ for the irreducible, finite dimensional $U_q(\mathfrak{k})$-module

$$V(\kappa, \kappa_1) = L_{(-\kappa_1)^n} \otimes L_{(\kappa_1 + (n-1)\kappa, (\kappa_1 - \kappa)^{n-1})}.$$

Here we have used the shorthand notation $(\kappa^m) \in P_m^+$ for the m-tuple with all entries equal to κ. The representation map of $V(\kappa, \kappa_1)$ will be denoted by $\rho(\kappa, \kappa_1)$. Clearly there exists a scalar product on $V(\kappa, \kappa_1)$, unique up to constant multiples, such that $\rho(\kappa, \kappa_1) : U_q(\mathfrak{k}) \to \mathrm{End}_{\mathbb{C}}(V(\kappa, \kappa_1))$ is a $*$-representation. Since $L_{(-\kappa_1)^n}$ is one-dimensional, $x_{n-1} \in U_q(\mathfrak{gl}(n))$ acts as zero, hence $I \subseteq \ker(\rho(\kappa, \kappa_1))$. Thus we obtain a $*$-representation

$$\rho(\kappa, \kappa_1)_\sigma : \mathcal{A}_\sigma \to \mathrm{End}_{\mathbb{C}}(V(\kappa, \kappa_1)_\sigma)$$

by the previous corollary. To simplify notations, we denote

$$(\rho(\kappa_1)_\sigma, V(\kappa_1)_\sigma) = (\rho(0, \kappa_1)_\sigma, V(0, \kappa_1)_\sigma).$$

The $U_q(\mathfrak{gl}(n))$-module $L_{((n-1)\kappa, (-\kappa)^{n-1})}$ is the quantum analogue of the $\mathrm{GL}(n; \mathbb{C})$-module $S^{n\kappa}(\mathbb{C}^n) \det^{-\kappa}$, where $S^m(\mathbb{C}^n)$ is the space of homogeneous polynomials on $\mathbb{C}^n$ of degree m. This is precisely the $U_q(\mathfrak{gl}(n))$-module which plays the crucial role in Etingof's and Kirillov's [3] theory on A-type Macdonald polynomials and generalized quantum group characters. The module $L_{((n-1)\kappa, (-\kappa)^{n-1})}$ admits a concrete description (see, e.g., [3, §5]) which can be directly lifted to $V(\kappa, \kappa_1)_\sigma$, leading to the following result.

Lemma 1.13. *Let $\mathcal{J}_\kappa$ be the set of n-tuples $\underline{m}=(m_1,\ldots,m_n)\in\mathbb{Z}_{\geq 0}^n$ which sum up to $n\kappa$. There exists a σ-independent basis $\{r_{\underline{m}}\,|\,\underline{m}\in\mathcal{J}_\kappa\}$ of $V(\kappa,\kappa_1)$ such that*

$$\begin{aligned}
\rho(\kappa,\kappa_1)_\sigma(C_i)r_{\underline{m}} &= q^{m_i-\kappa}r_{\underline{m}},\\
\rho(\kappa,\kappa_1)_\sigma(B_j)r_{\underline{m}} &= q^{-m_{j+1}+\kappa}\left(\frac{q^{m_j}-q^{-m_j}}{q-q^{-1}}\right)r_{\underline{m}-\varepsilon_j+\varepsilon_{j+1}},\\
\rho(\kappa,\kappa_1)_\sigma(B_{2n-j})r_{\underline{m}} &= q^{-m_j+\kappa}\left(\frac{q^{m_{j+1}}-q^{-m_{j+1}}}{q-q^{-1}}\right)r_{\underline{m}+\varepsilon_j-\varepsilon_{j+1}},\\
\rho(\kappa,\kappa_1)_\sigma(B_n^\sigma)r_{\underline{m}} &= \left(\frac{q^{-\sigma+2(\kappa-\kappa_1-m_n)}-q^{\sigma+2\kappa_1}}{q-q^{-1}}\right)r_{\underline{m}}
\end{aligned}$$

for $i=1,\ldots,n$ and $j=1,\ldots,n-1$, where we have used the convention that $r_{\underline{m}}=0$ if $\underline{m}\notin\mathcal{J}_\kappa$.

Lemma 1.13 implies that the common eigenspace

$$\widetilde{V}(\kappa,\kappa_1)_\sigma=\{v\in V(\kappa,\kappa_1)_\sigma\;\mid\;\rho(\kappa,\kappa_1)_\sigma(C_i)v=v\;\;\forall i\}\tag{1.15}$$

is one-dimensional and spanned by $r_{(\kappa^n)}$.

The following lemma follows directly from Lemma 1.13.

Lemma 1.14. *The $\mathcal{A}_\sigma$-module $V(\kappa,\kappa_1)_\sigma$, viewed as $U_q(\mathfrak{gl}(n))$-module by restricting the module structure to $\mathcal{E}\subset\mathcal{A}_\sigma$ and using the isomorphism $\mathcal{E}\simeq U_q(\mathfrak{gl}(n))$ of Lemma 1.8, is isomorphic to the simple $U_q(\mathfrak{gl}(n))$-module $L_{((n-1)\kappa,(-\kappa)^{n-1})}$. In particular, $V(\kappa,\kappa_1)_\sigma$ is a simple $\mathcal{A}_\sigma$-module.*

We end this subsection by formulating branching rules for the quantum symmetric pair $(U_q(\mathfrak{g}),\mathcal{A}_\sigma)$. For the formulation it is convenient to define an injective map $^\natural:\Lambda_n\to P_{2n}$ by

$$\mu^\natural=(\mu_1,\mu_2,\ldots,\mu_n,-\mu_n,\ldots,-\mu_2,-\mu_1).\tag{1.16}$$

Observe that $^\natural$ restricts to a map $^\natural:\Lambda_n^+\to P_{2n}^+$, and that $^\natural$ respects the dominance order, so $\mu^\natural\preceq\nu^\natural$ for $\mu,\nu\in\Lambda_n$ if $\mu\leq\nu$.

Proposition 1.15. *Let $\kappa_2\in\mathbb{Z}$ and $\kappa,\kappa_1\in\mathbb{Z}_{\geq 0}$ with $-\kappa_1\leq\kappa_2\leq\kappa_1$, and set*

$$\delta(\kappa,\kappa_1)=(\kappa_1+(n-1)\kappa,\kappa_1+(n-2)\kappa,\ldots,\kappa_1)\in\Lambda_n^+.\tag{1.17}$$

Let $\lambda\in P_{2n}^+$. The simple $\mathcal{A}_\sigma$-modules $V(\kappa,\kappa_1)_\sigma$ and $V(\kappa_2)_\sigma$ are both constituents of the $\mathcal{A}_\sigma$-module L_λ^σ if and only if $\lambda\in(\Lambda_n^++\delta(\kappa,\kappa_1))^\natural$. For such λ, both $V(\kappa,\kappa_1)_\sigma$ and $V(\kappa_2)_\sigma$ occur with multiplicity one in L_λ^σ.

For $\kappa=\kappa_1=\kappa_2=0$, Proposition 1.15 states which of the simple, finite dimensional $U_q(\mathfrak{g})$-modules L_λ are spherical with respect to $\mathcal{A}_\sigma$. This was proven for general quantum symmetric pairs in [13, Thm. 4.3]. This special case can also be derived from the results in [19], see Lemma 3.5. The classical case ($q=1$) of Proposition 1.15 was proven in [22] using the Littlewood-Richardson rule. Finally, for $n=1$ (in which case the parameter κ is reduntant), Proposition 1.15 is due to Koornwinder [9], who used q-special functions for the proof. The general proof of Proposition 1.15 is discussed in §2.

Remark 1.16. In the remainder of the paper we always assume the conditions $\kappa, \kappa_1 \in \mathbb{Z}_{\geq 0}$ and $-\kappa_1 \leq \kappa_2 \leq \kappa_1$ on the representation labels $\kappa, \kappa_1, \kappa_2 \in \mathbb{Z}$, and we use the short hand notation $\vec{\kappa} = (\kappa_1, \kappa_2, \kappa)$. There are three other parameter domains for $\vec{\kappa}$ for which similar results can be derived with only minor alterations, namely $\kappa_1, \kappa_2 \in \mathbb{Z}$, $\kappa \in \mathbb{Z}_{\geq 0}$ satisfying $\kappa_1 \in \mathbb{Z}_{\leq 0}$ and $\kappa_1 \leq \kappa_2 \leq -\kappa_1$, or $\kappa_2 \in \mathbb{Z}_{\geq 0}$ and $-\kappa_2 \leq \kappa_1 \leq \kappa_2$, or $\kappa_2 \in \mathbb{Z}_{\leq 0}$ and $\kappa_2 \leq \kappa_1 \leq -\kappa_2$, compare with [8] for the special case $n = 1$ and with [22] for the classical case ($q = 1$).

1.5. **Vector valued spherical functions.** We write $G = \mathrm{GL}(2n; \mathbb{C})$ for the general linear group. The quantized algebra of regular functions $\mathbb{C}_q[G] \subseteq U_q(\mathfrak{g})^*$ is the span of the matrix coefficients of the irreducible $U_q(\mathfrak{g})$-representations L_λ ($\lambda \in P_{2n}^+$). The quantized function algebra $\mathbb{C}_q[G]$ inherets from $U_q(\mathfrak{g})$ the structure of a Hopf $*$-algebra. In particular, the $*$-structure on $\mathbb{C}_q[G]$ is defined by

$$f^*(X) = \overline{f(S(X)^*)}, \qquad \forall\, X \in U_q(\mathfrak{g}).$$

The left and right regular $U_q(\mathfrak{g})$-action on $\mathbb{C}_q[G]$ is defined by

$$(Y \cdot f \cdot Z)(X) = f(ZXY), \qquad X, Y, Z \in U_q(\mathfrak{g}).$$

The Peter-Weyl decomposition

$$\mathbb{C}_q[G] = \bigoplus_{\lambda \in P_{2n}^+} W(\lambda) \tag{1.18}$$

with $W(\lambda)$ the span of the matrix coefficients of L_λ, is the irreducible decomposition of $\mathbb{C}_q[G]$ as $U_q(\mathfrak{g})$-bimodule. For any vector space V, we may and will view elements $f \in \mathbb{C}_q[G] \otimes V$ as linear maps $f : U_q(\mathfrak{g}) \to V$.

Recall the conventions and notations for the representation labels $\vec{\kappa}$ from Remark 1.16. We fix $\sigma, \tau \in \mathbb{R}$ once and for all, unless specified differently. In the following definition we identify $V(\kappa_2)_\sigma \simeq \mathbb{C}$ as vector spaces and view $\rho(\kappa_2)_\sigma$ as character of $\mathcal{A}_\sigma$.

Definition 1.17. *We call $f \in \mathbb{C}_q[G] \otimes V(\kappa, \kappa_1)_\tau$ a vector valued spherical function if for all $X \in U_q(\mathfrak{g})$,*

$$\begin{aligned} f(Xa) &= \rho(\kappa_2)_\sigma(a)\, f(X), \qquad \forall\, a \in \mathcal{A}_\sigma,\\ f(bX) &= \rho(\kappa, \kappa_1)_\tau(b)\, f(X), \qquad \forall\, b \in \mathcal{A}_\tau. \end{aligned}$$

We denote by $F_{\vec{\kappa}}^{\sigma,\tau}$ the space of vector valued spherical functions.

By standard arguments (compare, e.g., with [22] for the classical setup, and with Lemma 4.1), we obtain the following corollary of Lemma 1.10 and Proposition 1.15.

Corollary 1.18. *The space of vector valued spherical functions $F_{\vec{\kappa}}^{\sigma,\tau}$ decomposes as*

$$F_{\vec{\kappa}}^{\sigma,\tau} = \bigoplus_{\mu \in \Lambda_n^+} F_{\vec{\kappa}}^{\sigma,\tau}(\mu),$$

with $F_{\vec{\kappa}}^{\sigma,\tau}(\mu)$ the one-dimensional vector space

$$F_{\vec{\kappa}}^{\sigma,\tau}(\mu) = F_{\vec{\kappa}}^{\sigma,\tau} \cap \Big(W\big((\mu + \delta(\kappa, \kappa_1))^\natural\big) \otimes V(\kappa, \kappa_1)_\tau\Big), \qquad \mu \in \Lambda_n^+.$$

We call a function $0 \neq f \in F^{\sigma,\tau}_{\vec{\kappa}}(\mu)$ an *elementary* vector valued spherical function of degree $\mu \in \Lambda_n^+$.

We consider now the restriction of vector valued spherical functions to the commutative subalgebra $U^0 \subset U_q(\mathfrak{g})$ generated by the group-like elements $K_j^{\pm 1}$ $(j = 1, \ldots, 2n)$. Denote

$$K^\lambda = K_1^{\lambda_1} K_2^{\lambda_2} \cdots K_{2n}^{\lambda_{2n}}, \qquad \lambda \in P_{2n},$$

which form a linear basis of U^0, and define $\delta = \delta_{2n} \in P_{2n}^+$ by

$$\delta = (2n-1, 2n-2, \ldots, 1, 0).$$

Let $\mathbb{C}[z^{\pm 1}]$ be the algebra of Laurent polynomials in $2n$ variables $z_1, \ldots, z_{2n}$.

Definition 1.19. *Let V be a complex vector space and $f \in \mathbb{C}_q[G] \otimes V$, viewed as linear map $f : U_q(\mathfrak{g}) \to V$. We define a regular function $f|_T : (\mathbb{C}^*)^{2n} \to V$ (or, equivalently, $f|_T \in \mathbb{C}[z^{\pm 1}] \otimes V$) by the requirement that*

$$f|_T(q^\lambda) = f(K^{\lambda - \delta}) \qquad \forall\, \lambda \in P_{2n},$$

where $q^\lambda = (q^{\lambda_1}, \ldots, q^{\lambda_{2n}})$. The resulting linear map

$$|_T : \mathbb{C}_q[G] \otimes V \to \mathbb{C}[z^{\pm 1}] \otimes V, \qquad f \mapsto f|_T$$

is called the restriction map.

We define elements $u_i \in \mathbb{C}[z^{\pm 1}]$ by

$$u_i = z_i z_{2n+1-i}^{-1}, \qquad i = 1, \ldots, n, \tag{1.19}$$

and we write $\mathbb{C}[u^{\pm 1}] \subset \mathbb{C}[z^{\pm 1}]$ for the subalgebra generated by the $u_i^{\pm 1}$. Recall the one-dimensional subspace $\widetilde{V}(\kappa, \kappa_1)_\tau \subseteq V(\kappa, \kappa_1)_\tau$ defined by (1.15).

Lemma 1.20. *If $f \in F^{\sigma,\tau}_{\vec{\kappa}} \subset \mathbb{C}_q[G] \otimes V(\kappa, \kappa_1)_\tau$, then $f|_T \in \mathbb{C}[u^{\pm 1}] \otimes \widetilde{V}(\kappa, \kappa_1)_\tau$. Identifying the one-dimensional vector space $\widetilde{V}(\kappa, \kappa_1)_\tau$ with $\mathbb{C}$, the restriction map thus gives rise to a linear map*

$$|_T : F^{\sigma,\tau}_{\vec{\kappa}} \to \mathbb{C}[u^{\pm 1}].$$

Proof. Fix $f \in F^{\sigma,\tau}_{\vec{\kappa}}$. For $i = 1, \ldots, n$ and $X \in U^0$ we compute

$$\begin{aligned} \rho(\kappa, \kappa_1)_\tau(C_i) f(X) &= f(C_i X) \\ &= f(X C_i) \\ &= \rho(\kappa_2)_\sigma(C_i) f(X) = f(X), \end{aligned}$$

hence $f(X) \in \widetilde{V}(\kappa, \kappa_1)_\tau$. Identifying $\widetilde{V}(\kappa, \kappa_1)_\tau \simeq \mathbb{C}$, we thus conclude that $f|_T \in \mathbb{C}[z^{\pm 1}]$. The formulas $f(XC_i) = f(X)$ for $X \in U^0$ and $i = 1, \ldots, n$ imply that $f|_T \in \mathbb{C}[u^{\pm 1}]$. $\square$

The following main result of this paper gives a representation theoretic interpretation of the full five parameter family of Macdonald-Koornwinder polynomials (with two parameters continuous, and three parameters discrete).

Theorem 1.21. *Fix parameters $\sigma,\tau\in\mathbb{R}$ and $\vec{\kappa}=(\kappa_1,\kappa_2,\kappa)\in\mathbb{Z}^{\times 3}$ with $\kappa,\kappa_1\in\mathbb{Z}_{\geq 0}$ and $-\kappa_1\leq\kappa_2\leq\kappa_1$. Let $f_0\in F_{\vec{\kappa}}^{\sigma,\tau}(0)$ be an elementary vector valued spherical function of degree $0\in\Lambda_n^+$.*

(i) *The restriction map $|_T$ defines a linear bijection*

$$|_T : F_{\vec{\kappa}}^{\sigma,\tau}\rightarrow f_0|_T\,\mathbb{C}[u^{\pm 1}]^W.$$

(ii) *We have*

$$\begin{aligned} f_0|_T = C\,u^{\delta(\kappa,\kappa_1)}\prod_{i=1}^n\bigl(q^{1-\sigma+\tau}u_i^{-1};q^2\bigr)_{\kappa_1-\kappa_2}\bigl(-q^{1+\sigma+\tau}u_i^{-1};q^2\bigr)_{\kappa_1+\kappa_2}\\ \times\prod_{1\leq i<j\leq n}\bigl(q^2u_i^{-1}u_j,q^2u_i^{-1}u_j^{-1};q^2\bigr)_\kappa \end{aligned}$$

for some nonzero constant C.

(iii) *If $f_\mu\in F_{\vec{\kappa}}^{\sigma,\tau}(\mu)$ is an elementary vector valued spherical function of degree $\mu\in\Lambda_n^+$, then*

$$\frac{f_\mu|_T}{f_0|_T}=D\,P_\mu\bigl(u;-q^{\sigma+\tau+1+\kappa_1+\kappa_2},-q^{-\sigma-\tau+1},q^{\sigma-\tau+1},q^{-\sigma+\tau+1+\kappa_1-\kappa_2};q^2,q^{2\kappa+2}\bigr)$$

for some nonzero constant D.

Remark 1.22. In the special case $n=1$ Theorem 1.21 gives a representation theoretic interpretation of the Askey-Wilson polynomials, which is in accordance with the results from [9], [20] and [8]. The classical case ($q=1$) of Theorem 1.21 was proven in [22]. In §3 we show that Theorem 1.21 for $\kappa_1=\kappa_2=\kappa=0$ follows from the main results in [19].

The proof of Theorem 1.21**(i)**, **(ii)** and **(iii)** is given in §4, §5 and §6, respectively.

2. Branching rules

The main goal of this section is to establish the specific branching rules for the quantum symmetric pairs $(U_q(\mathfrak{g}),\mathcal{A}_\sigma)$ as formulated in Proposition 1.15. We start with defining and studying the notion of the expectation value for the quantum symmetric pairs $(U_q(\mathfrak{g}),\mathcal{A}_\sigma)$.

2.1. **The expectation value.** The expectation value for intertwiners plays a crucial role in the representation theoretic approach to quantum field theory, see, e.g., [4]. The notion of expectation value is naturally associated to symmetric pairs $(H\times H,\mathrm{diag}(H))$ with H a semisimple Lie group and $\mathrm{diag}(H)$ the diagonal embedding of H in $H\times H$, as well as to their quantum analogues $(U_q(\mathfrak{h})\otimes U_q(\mathfrak{h}),\Delta(U_q(\mathfrak{h})))$ with $\mathfrak{h}$ a semisimple Lie algebra. In this subsection we define the expectation value for the quantum symmetric pair $(U_q(\mathfrak{g}),\mathcal{A}_\sigma)$. As we shall see, this directly leads to information on branching rules for the quantum symmetric pairs $(U_q(\mathfrak{g}),\mathcal{A}_\sigma)$.

For $\lambda\in P_{2n}^+$ we choose a highest weight vector $0\neq v_\lambda\in L_\lambda[\lambda]$ and a lowest weight vector $0\neq v_{w_0\lambda}\in L_\lambda[w_0\lambda]$, where $w_0\in S_{2n}$ is the longest Weyl group element, $w_0(i)=2n+1-i$ for all i. Both v_λ and $v_{w_0\lambda}$ are unique up to a multiplicative constant.

Definition 2.1. *Let M be a left $\mathcal{A}_\sigma$-module and denote $Hom_{\mathcal{A}_\sigma}(L^\sigma_\lambda, M)$ for the space of $\mathcal{A}_\sigma$-intertwiners $L^\sigma_\lambda \to M$. The maps $h^\lambda_M, l^\lambda_M : Hom_{\mathcal{A}_\sigma}(L^\sigma_\lambda, M) \to M$, defined by $h^\lambda_M(\phi) = \phi(v_\lambda)$ and $l^\lambda_M(\phi) = \phi(v_{w_0\lambda})$, are called the higher and lower expectation maps, respectively. The values $h^\lambda_M(\phi)$ and $l^\lambda_M(\phi)$ are called the higher and lower expectation value of ϕ.*

Define a map $\flat : P_{2n} \to \Lambda_n$ by

$$\lambda^\flat = (\lambda_1 + \lambda_{2n}, \lambda_2 + \lambda_{2n-1}, \cdots, \lambda_n + \lambda_{n+1}), \qquad \lambda \in P_{2n}.$$

Observe that the kernel of $\flat$ equals $\Lambda^\natural_n$ and that $(w_0\lambda)^\flat = \lambda^\flat$ for all $\lambda \in P_{2n}$. If M is a left $\mathcal{A}_\sigma$-module and $\mu \in \Lambda_n$, then we define

$$M_\mu = \{m \in M \,|\, C_i m = q^{\mu_i} m, \quad \forall\, i = 1, \dots, n\}. \tag{2.1}$$

Proposition 2.2. *Let M be a left $\mathcal{A}_\sigma$-module and let $\lambda \in P^+_{2n}$. The expectation maps h^λ_M and l^λ_M are injective, with image contained in $M_{\lambda^\flat}$.*

Proof. We prove the proposition for the expectation map h^λ_M, the proof for l^λ_M is analogous. For $i = 1, \dots, n$ and $\phi \in \mathrm{Hom}_{\mathcal{A}_\sigma}(L^\sigma_\lambda, M)$ we have

$$C_i h^\lambda_M(\phi) = C_i \phi(v_\lambda) = \phi(C_i v_\lambda) = q^{\lambda_i + \lambda_{2n+1-i}} h^\lambda_M(\phi),$$

so $h^\lambda_M(\phi) \in M_{\lambda^\flat}$. Thus the image of h^λ_M is contained in $M_{\lambda^\flat}$.

Suppose $\phi \in \mathrm{Hom}(L^\sigma_\lambda, M)$ is in the kernel of h^λ_M, so $\phi(v_\lambda) = 0$. We show that $\phi(v) = 0$ for $v \in L_\lambda[\nu]$ and $\nu \preceq \lambda$ by induction on the height $\mathrm{ht}(\lambda - \nu) \in \mathbb{Z}_{\geq 0}$, where $\mathrm{ht}(\lambda - \nu) = \sum_{i=1}^{2n-1} m_i$ if m_i are the unique positive integers such that $\lambda - \nu = \sum_i m_i \alpha_i$. So suppose that $\phi(v) = 0$ for all vectors $v \in L_\lambda[\nu]$ with $\nu \preceq \lambda$ and $\mathrm{ht}(\lambda - \nu) \leq N$.

For arbitrary $\nu \in P_{2n}$ we have $L_\lambda[\nu] = \{0\}$ unless $\nu \preceq \lambda$ and

$$L_\lambda[\nu] = \sum_{j=1}^{2n-1} \widetilde{y}_j L_\lambda[\nu + \alpha_j],$$

where $\widetilde{y}_j = y_j K^{-1}_{j+1} K^{-1}_{2n-j}$. To prove the induction step, it thus suffices to show that $\phi(\widetilde{y}_j v) = 0$ for all j when $v \in L_\lambda[\nu]$, $\nu \preceq \lambda$ and $\mathrm{ht}(\lambda - \nu) = N$. If v is such a vector, then by the explicit form of B_j and B^σ_n and by the induction hypothesis,

$$\phi(\widetilde{y}_j v) = \phi(B_j v) = B_j \phi(v) = 0,$$
$$\phi(\widetilde{y}_n v) = \phi(B^\sigma_n v) = B^\sigma_n \phi(v) = 0$$

for $j \in \{1, \dots, 2n-1\} \setminus \{n\}$, as desired. □

Since L^σ_λ is $\mathcal{A}_\sigma$-semisimple by Lemma 1.10, the previous proposition has the following immediate consequence.

Corollary 2.3. *If M is a simple $\mathcal{A}_\sigma$-module and $\lambda \in P^+_{2n}$, then the number of summands isomorphic to M in the irreducible decomposition of L^σ_λ is bounded by $Dim(M_{\lambda^\flat})$.*

Recall the conditions on the representation labels $\vec{\kappa} = (\kappa_1, \kappa_2, \kappa)$ from Remark 1.16.

Lemma 2.4. *Let $\lambda \in P_{2n}^+$. If both $V(\kappa,\kappa_1)_\sigma$ and $V(\kappa_2)_\sigma$ occur as constituent in the irreducible decomposition of the $\mathcal{A}_\sigma$-semisimple module L_λ^σ, then $\lambda \in (\Lambda_n^+ + \delta(\kappa,\kappa_1))^\natural$. For such λ, the modules $V(\kappa,\kappa_1)_\sigma$ and $V(\kappa_2)_\sigma$ occur at most with multiplicity one in L_λ^σ.*

Proof. The weight spaces M_μ (see (2.1)) for the simple $\mathcal{A}_\sigma$-modules $M = V(\kappa,\kappa_1)_\sigma$ and $M = V(\kappa_2)_\sigma$ are at most one-dimensional by Lemma 1.13. Thus for any $\lambda \in P_{2n}^+$, the number of summands isomorphic to $V(\kappa,\kappa_1)_\sigma$ or to $V(\kappa_2)_\sigma$ in the irreducible decomposition of L_λ^σ is at most one by Corollary 2.3.

If $v \in V(\kappa_2)_\sigma$ then $C_i v = v$ for all i, hence $V(\kappa_2)_\sigma$ can only occur as constituent in the irreducible decomposition of L_λ^σ when $\lambda \in P_{2n}^+ \cap \mathrm{Ker}(\flat) = (\Lambda_n^+)^\natural$.

Fix $\mu \in \Lambda_n^+$ such that $V(\kappa,\kappa_1)_\sigma$ occurs as constituent of the irreducible decomposition of $L_{\mu^\natural}^\sigma$. It remains to show that this assumption imposes the additional restrictions $\mu_n \geq \kappa_1$ and $\mu_j - \mu_{j+1} \geq \kappa$ ($j = 1,\ldots,n-1$) on μ. Throughout the proof, we fix a nonzero intertwiner $\phi \in \mathrm{Hom}_{\mathcal{A}_\sigma}\big(L_{\mu^\natural}^\sigma, V(\kappa,\kappa_1)_\sigma\big)$.

We start with proving $\mu_n \geq \kappa_1$. We use the notations of Remark 1.7. Standard $U_q(\mathfrak{gl}(2))$-representation theory implies that $V = U_q(\mathfrak{gl}(2))^{(n)} v_{\mu^\natural} \subseteq L_{\mu^\natural}$ is the $(2\mu_n+1)$-dimensional irreducible representation of $U_q(\mathfrak{gl}(2))^{(n)} \simeq U_q(\mathfrak{gl}(2))$ with highest weight $(\mu_n,-\mu_n)$ and highest weight vector $v_{\mu^\natural}$, and that $\widehat{B}_n^\sigma|_V \in \mathrm{End}_{\mathbb{C}}(V)$ is semisimple with simple spectrum

$$s_l = \frac{q^{-\sigma+2l} - q^{\sigma-2l}}{q-q^{-1}}, \qquad l = -\mu_n,\ldots,\mu_n - 1,\mu_n.$$

Observe furthermore that V is x_σ-stable. Since $x_\sigma B_n^\sigma x_\sigma^{-1} = \widehat{B}_n^\sigma$, it follows that $B_n^\sigma|_V \in \mathrm{End}_{\mathbb{C}}(V)$ is semisimple with simple spectrum $\{s_l \,|\, l = -\mu_n,\ldots,\mu_n-1,\mu_n\}$. Proposition 2.2 and the fact that C_i centralizes $U_q(\mathfrak{gl}(2))^{(n)}$ for $i = 1,\ldots,n$ imply that $\phi(V) = \widetilde{V}(\kappa,\kappa_1)_\sigma$ with $\widetilde{V}(\kappa,\kappa_1)_\sigma$ the one-dimensional space defined by (1.15). On the other hand, Lemma 1.13 shows that $\widetilde{V}(\kappa,\kappa_1)_\sigma$ is the eigenspace of $\rho(\kappa,\kappa_1)_\sigma(B_n^\sigma)$ with eigenvalue $s_{-\kappa_1}$, hence $s_{-\kappa_1}$ must be in the spectrum of $B_n^\sigma|_V$. Consequently, $\mu_n \geq \kappa_1$.

Next we show that $\mu_j - \mu_{j+1} \geq \kappa$ for $j = 1,\ldots,n-1$. Fix $j \in \{1,\ldots,n-1\}$. Since $0 \neq \phi(v_{\mu^\natural}) \in \widetilde{V}(\kappa,\kappa_1)_\sigma$ by Proposition 2.2, we conclude from the explicit description of $V(\kappa,\kappa_1)_\sigma$ (see Lemma 1.13) that

$$\phi(B_j^\kappa v_{\mu^\natural}) = \rho(\kappa,\kappa_1)_\sigma(B_j^\kappa)\phi(v_{\mu^\natural}) \neq 0.$$

This implies that $B_j^\kappa v_{\mu^\natural} \neq 0$ in $L_{\mu^\natural}$. We write $B_j = \widetilde{y}_j + \widetilde{x}_{2n-j}$ with

$$\widetilde{y}_j = y_j K_{j+1}^{-1} K_{2n-j}^{-1}, \qquad \widetilde{x}_{2n-j} = K_j^{-1} x_{2n-j} K_{2n-j}^{-1}.$$

Since $v_{\mu^\natural} \in L_{\mu^\natural}$ is a highest weight vector, the commutation relation $\widetilde{x}_{2n-j}\widetilde{y}_j = q^2 \widetilde{y}_j \widetilde{x}_{2n-j}$ in $U_q(\mathfrak{g})$ implies

$$\widetilde{y}_j^\kappa v_{\mu^\natural} = B_j^\kappa v_{\mu^\natural} \neq 0,$$

and consequently $y_j^\kappa v_{\mu^\natural} \neq 0$. Standard arguments from $U_q(\mathfrak{gl}(2)) \simeq \mathbb{C}\langle x_j, y_j, K_j^{\pm 1}, K_{j+1}^{\pm 1}\rangle$ representation theory imply that $\mu_j - \mu_{j+1}$ is the largest positive integer m for which $y_j^m v_{\mu^\natural} \neq 0$ in $L_{\mu^\natural}$. We conclude that $\mu_j - \mu_{j+1} \geq \kappa$, as desired. □

In, e.g., [18] and [19] highest weight considerations are used to determine branching rules for quantum symmetric pairs. These highest weight arguments can be formalized and generalized in the present setting as follows.

Lemma 2.5. *Let $\lambda \in P_{2n}^+$ and let $\{0\} \neq M \subseteq L_\lambda^\sigma$ be an $\mathcal{A}_\sigma$-submodule. Then*

$$M \not\subseteq \bigoplus_{\nu \prec \lambda} L_\lambda[\nu], \qquad M \not\subseteq \bigoplus_{\nu \succ w_0\lambda} L_\lambda[\nu].$$

In other words, there exists a vector $m_+ \in M$ (resp. $m_- \in M$) whose decomposition in U^0-weights has a nonzero highest (resp. lowest) weight contribution.

Proof. Suppose $M \subseteq L_\lambda^\sigma$ is an $\mathcal{A}_\sigma$-submodule and

$$M \subseteq \bigoplus_{\nu \prec \lambda} L_\lambda[\nu]. \tag{2.2}$$

Let $M^\perp$ be the orthocomplement of M in L_λ with respect to the $*$-unitary scalar product $\langle \cdot, \cdot \rangle_\lambda$ on L_λ. Then $M^\perp$ is an $\mathcal{A}_\sigma$-module complement of M in L_λ^σ. Let $\pi \in \mathrm{Hom}_{\mathcal{A}_\sigma}(L_\lambda^\sigma, M)$ be the projection onto M along $M^\perp$. By the $*$-selfadjointness of the K_i's, we have $L_\lambda[\nu] \perp L_\lambda[\nu']$ for $\nu \neq \nu'$. Condition (2.2) thus implies $L_\lambda[\lambda] \subseteq M^\perp$. Hence the higher expectation value $h_M^\lambda(\pi)$ is zero. Proposition 2.2 then implies $\pi \equiv 0$, hence $M = \{0\}$. The second statement is proved similarly. □

Remark 2.6. For an arbitrary quantum symmetric pairs (U, B_θ) in the sense of Letzter (see [15, §7] for a complete list), one can make a similar definition of higher and lower expectation maps and of expectation values. Following the reasoning of this section, one can establish the analogues of Proposition 2.2, Corollary 2.3 and Lemma 2.5 in this general setup (cf. the proof of [16, Lem. 5.6]).

2.2. **Deformation arguments.** In this subsection we complete the proof of Proposition 1.15 by using deformation arguments. Without loss of generality we may take $q = e^h$ to be a formal deformation parameter. We use part four of Kassel's book [6] as main reference for facts and notations related to topological $\mathbb{C}[[h]]$-modules and topological $\mathbb{C}[[h]]$-algebras.

Let $U_h(\mathfrak{g})$ be the topological version of the quantized universal enveloping algebra $U_q(\mathfrak{g})$, with topological generators $E_{i,i}$, x_j and y_j ($i \in \{1, \ldots, 2n\}$, $j \in \{1, \ldots, 2n-1\}$), where $K_i = e^{hE_{i,i}}$. Let $U_h(\mathfrak{sl}(2n))$ be the subalgebra of $U_h(\mathfrak{gl}(2n))$ topologically generated by the elements $E_{j,j} - E_{j+1,j+1}$, x_j and y_j ($j \in \{1, \ldots, 2n-1\}$). Note that $U_h(\mathfrak{g})$ is the central extension of $U_h(\mathfrak{sl}(2n))$ by the central element $E_{1,1} + E_{2,2} + \cdots + E_{2n,2n}$.

The topological version of $\mathcal{A}_\sigma$ is the subalgebra of $U_h(\mathfrak{g})$ topologically generated by B_j, B_n^σ and $E_{i,i} + E_{2n+1-i,2n+1-i}$ for $j \in \{1, \ldots, 2n-1\} \setminus \{n\}$ and $i \in \{1, \ldots, n\}$. We denote by $\overline{\mathcal{A}}_\sigma$ the subalgebra of $U_h(\mathfrak{sl}(2n))$ topologically generated by B_jC_{j+1}, $B_n^\sigma C_n$ and $E_{i,i} - E_{i+1,i+1} - E_{2n-i,2n-i} + E_{2n+1-i,2n+1-i}$ for $j \in \{1, \ldots, 2n-1\} \setminus \{n\}$ and $i \in \{1, \ldots, n-1\}$. Note that $\mathcal{A}_\sigma$ is the central extension of $\overline{\mathcal{A}}_\sigma$ by the central element $E_{1,1} + E_{2,2} + \cdots + E_{2n,2n} \in U_h(\mathfrak{g})$.

We fix $\lambda = \big(\mu + \delta(\kappa, \kappa_1)\big)^\natural$ with $\mu \in \Lambda_n^+$. To complete the proof of Proposition 1.15, it suffices to show that the irreducible $\mathcal{A}_\sigma$-module $V(\kappa, \kappa_1)_\sigma$ occurs as consituent of L_λ^σ.

Since the central element $E_{1,1}+\cdots+E_{2n,2n}$ acts as zero on the topological versions of $V(\kappa,\kappa_1)_\sigma$ and L_λ, it suffices to show that $V(\kappa,\kappa_1)_\sigma$ is a constituent of L_λ^σ when viewed as $\overline{\mathcal{A}}_\sigma$-modules.

The formal computations after Definition 1.5 and [14, Thm. 7.5] imply that the algebra $\overline{\mathcal{A}}_\sigma$ is a deformation of the universal enveloping algebra $U(\overline{\mathfrak{k}})$ with

$$\overline{\mathfrak{k}}=\mathrm{Ad}(g)\bigl(\mathfrak{k}\bigr)\cap\mathfrak{sl}(2n)=\overline{\mathfrak{k}}_{ss}\oplus\mathbb{C}Z,$$

where g is given by (1.11), $\overline{\mathfrak{k}}_{ss}=\mathrm{Ad}(g)\bigl(\mathfrak{sl}(n)\oplus\mathfrak{sl}(n)\bigr)$, and $Z\in\overline{\mathfrak{k}}$ the central element

$$Z=\mathrm{Ad}(g)(Z')=-\frac{1}{2n}\sum_{j=1}^{2n}E_{j,2n+1-j}$$

where

$$Z'=-\frac{1}{2n}\bigl(E_{1,1}+E_{2,2}+\cdots+E_{n,n}-E_{n+1,n+1}-E_{n+2,n+2}-\cdots-E_{2n,2n}\bigr).$$

Since $\overline{\mathfrak{k}}$ is reductive with one-dimensional center, we have $H^2(\overline{\mathfrak{k}},U(\overline{\mathfrak{k}}))=\{0\}$. By [6, Thm. XVIII.2.2] there exists a topological algebra isomorphism

$$\beta_\sigma:\overline{\mathcal{A}}_\sigma\to U(\overline{\mathfrak{k}})[[h]]$$

which is the identity modulo h. We define

$$\overline{\mathcal{A}}_\sigma^{ss}=\beta_\sigma^{-1}\bigl(U(\overline{\mathfrak{k}}_{ss})[[h]]\bigr),\qquad Z_\sigma=\beta_\sigma^{-1}(Z),$$

then $\overline{\mathcal{A}}_\sigma^{ss}$ is a deformation of $U(\overline{\mathfrak{k}}_{ss})$, Z_σ is central in $\overline{\mathcal{A}}_\sigma$, and $\overline{\mathcal{A}}_\sigma$ is topologically generated by $\overline{\mathcal{A}}_\sigma^{ss}$ and Z_σ. We denote by $(\beta_\sigma^{-1})^*(V(\kappa,\kappa_1)_\sigma)$ the module $V(\kappa,\kappa_1)_\sigma$ viewed as module over $U(\overline{\mathfrak{k}})[[h]]$ via the isomorphism $\beta_\sigma:\overline{\mathcal{A}}_\sigma\to U(\overline{\mathfrak{k}})[[h]]$ (we will use similar notations to indicate the pull back of an action by some algebra morpism). The way we constructed the module $V(\kappa,\kappa_1)_\sigma$ in §1 immediately implies that modulo h, we reobtain the irreducible, finite dimensional $\bigl(\mathfrak{gl}(n)\times\mathfrak{gl}(n)\bigr)\cap\mathfrak{sl}(2n)\simeq\overline{\mathfrak{k}}$ module $V^{clas}(\kappa,\kappa_1)$ of highest weight

$$\bigl((-\kappa_1)^n,\bigl(\kappa_1+(n-1)\kappa,(\kappa_1-\kappa)^{n-1}\bigr)\bigr)\in P_n^+\times P_n^+. \tag{2.3}$$

Observe that the central element Z acts as multiplication by κ_1 on $V^{clas}(\kappa,\kappa_1)$, while $V^{clas}(\kappa,\kappa_1)$ is independent of κ_1 when viewed as module over $\overline{\mathfrak{k}}_{ss}$. Since $\overline{\mathfrak{k}}_{ss}$ is semisimple, we conclude that $(\beta_\sigma^{-1})^*(V(\kappa,\kappa_1)_\sigma)$, viewed as $U(\overline{\mathfrak{k}}_{ss})[[h]]$-module, is isomorphic to the $\mathbb{C}[[h]]$-linear extension of the $\overline{\mathfrak{k}}_{ss}$-module $V^{clas}(\kappa,\kappa_1)$.

Lemma 2.7. *There exists a $\overline{\mathcal{A}}_\sigma$-submodule of $\overline{V}(\kappa,\kappa_1)_\sigma\subset L_\lambda^\sigma$ such that*
(i) $\overline{V}(\kappa,\kappa_1)_\sigma\simeq V(\kappa,\kappa_1)_\sigma$ *as $\overline{\mathcal{A}}_\sigma^{ss}$-modules.*
(ii) *The central element Z_σ acts as multiplication by a scalar $\overline{\xi}_\sigma\in\mathbb{C}[[h]]$ on $\overline{V}(\kappa,\kappa_1)_\sigma$.*
(iii) *Modulo h, $\overline{V}(\kappa,\kappa_1)_\sigma$ is the $\overline{\mathfrak{k}}$-module $V^{clas}(\kappa,\kappa_1)$.*

Proof. Since $U_h(\mathfrak{sl}(2n))$ is a deformation of the universal enveloping algebra of the simple Lie algebra $\mathfrak{sl}(2n)$, there exists by [6, Thm. XVIII.2.2] a topological algebra isomorphism

$$\alpha:U_h(\mathfrak{sl}(2n))\to U(\mathfrak{sl}(2n))[[h]]$$

which is the identity modulo h. The natural embedding $\iota_\sigma : \overline{\mathcal{A}}_\sigma \to U_h(\mathfrak{sl}(2n))$ of topological algebras can thus be pulled back to obtain an embedding

$$\gamma_\sigma = \alpha \circ \iota_\sigma \circ \beta_\sigma^{-1} : U(\overline{\mathfrak{k}})[[h]] \to U(\mathfrak{sl}(2n))[[h]]$$

of topological algebras, which is the identity modulo h. Since $\overline{\mathfrak{k}}_{ss}$ is semisimple, we have $H^1(\overline{\mathfrak{k}}_{ss}, U(\mathfrak{sl}(2n))) = \{0\}$, hence there exists an invertible element $F_\sigma \in U(\mathfrak{sl}(2n))[[h]]$ which is 1 modulo h such that

$$\gamma_\sigma(X) = F_\sigma X F_\sigma^{-1} \qquad \forall X \in U(\overline{\mathfrak{k}}_{ss})[[h]] \subset U(\overline{\mathfrak{k}})[[h]] \subset U(\mathfrak{sl}(2n))[[h]] \tag{2.4}$$

by [6, Thm. XVIII.2.1].

Let L_λ^{clas} be the irreducible $U(\mathfrak{sl}(2n))$- module corresponding to the highest weight λ. We denote $V \subset L_\lambda^{clas}$ for its $V^{clas}(\kappa, \kappa_1)$-isotypical component viewed as $U(\overline{\mathfrak{k}}_{ss})$-module. The central element $Z \in \overline{\mathfrak{k}}$ acts semisimply on V and the corresponding Z-eigenspaces V_r of V (with r the Z-eigenvalue) yield the $V^{clas}(\kappa, r)$ -isotypical components of L_λ^{clas} viewed now as $\overline{\mathfrak{k}}$-module. By the classical branching rules [22, Lemma 3] for the symmetric pair $(\mathfrak{g}, \mathfrak{gl}(n) \times \mathfrak{gl}(n))$ we have $V_{\kappa_1} \simeq V^{clas}(\kappa, \kappa_1)$.

Let $L_\lambda^{clas}[[h]]$ be the $U(\mathfrak{sl}(2n))[[h]]$-module obtained by $\mathbb{C}[[h]]$-linear extension of the $U(\mathfrak{sl}(2n))$-module structure on L_λ^{clas}. Then $L_\lambda^\sigma \simeq (\alpha \circ \iota_\sigma)^*(L_\lambda^{clas}[[h]])$ as topological $\overline{\mathcal{A}}_\sigma$-module. Now consider the $U(\overline{\mathfrak{k}})[[h]]$-module

$$(\beta_\sigma^{-1})^*(L_\lambda^\sigma) = \gamma_\sigma^*(L_\lambda^{clas}[[h]]).$$

By (2.4), $F_\sigma V[[h]]$ is the $V^{clas}(\kappa, \kappa_1)[[h]]$-isotypical component of the $U(\overline{\mathfrak{k}}_{ss})[[h]]$-module $\gamma_\sigma^*(L_\lambda^{clas}[[h]])$. The proof is now completed by observing that $F_\sigma V[[h]]$ contains a nonzero Z-eigenspace $\overline{V}(\kappa, \kappa_1)_\sigma$ for some eigenvalue $\overline{\xi}_\sigma \in \mathbb{C}[[h]]$ satisfying $\overline{\xi}_\sigma = \kappa_1$ modulo h. □

The central element $Z_\sigma \in \overline{\mathcal{A}}_\sigma$ acts by a scalar $\xi_\sigma \in \mathbb{C}[[h]]$ on $V(\kappa, \kappa_1)_\sigma$. To complete the arguments it thus remains to prove the following lemma.

Lemma 2.8. *$\xi_\sigma = \overline{\xi}_\sigma$ in $\mathbb{C}[[h]]$.*

Proof. We use the fact that $B_n^\sigma C_n \in \overline{\mathcal{A}}_\sigma$ is conjugate to the Cartan type element

$$\overline{B}_n^\sigma = \frac{q^{-\sigma} K_n K_{n+1}^{-1} - q^\sigma K_n^{-1} K_{n+1}}{q - q^{-1}}$$

in $U_h(\mathfrak{sl}(2n))$, see [23] and Remark 1.7. We first show that $B_n^\sigma C_n$ acts semisimply on $\overline{V}(\kappa, \kappa_1)_\sigma$, with spectrum contained in

$$\left\{ t_l = \frac{q^{-\sigma+l} - q^{\sigma-l}}{q - q^{-1}} \mid l \in \mathbb{Z} \right\}.$$

Indeed, $B_n^\sigma C_n$ acts semisimply on L_λ^σ, with spectrum contained in $\{t_l\}_{l \in \mathbb{Z}}$. Hence any vectors $v \in \overline{V}(\kappa, \kappa_1)_\sigma$ can be written as $v = \sum_l v_l$ with $v_l \in \ker(B_n^\sigma C_n - t_l) \subseteq L_\lambda^\sigma$. Since $B_n^\sigma C_n$ preserves $\overline{V}(\kappa, \kappa_1)_\sigma$, we have $(B_n^\sigma C_n)^k(v) = \sum_l t_l^k v_l \in \overline{V}(\kappa, \kappa_1)_\sigma$ for all $k \in \mathbb{Z}_{\geq 0}$. Simple calculations involving the Vandermonde determinant shows that there exists $c_k^l \in \mathbb{C}[[h]]$ such that $v_l = \sum_k c_k^l (B_n^\sigma C_n)^k v$ for all l, hence $v_l \in \overline{V}(\kappa, \kappa_1)_\sigma$ for all l. By Lemma 1.13, the same statement about the spectrum is valid for the action of $B_n^\sigma C_n$ on $V(\kappa, \kappa_1)_\sigma$.

Lemma 2.7 and the fact that the eigenvalues $t_l|_{h=0} = \sigma - l$ ($l \in \mathbb{Z}$) remain seperated modulo h imply that the dimensions over $\mathbb{C}[[h]]$ of the $B_n^\sigma C_n$-eigenspaces of $V(\kappa,\kappa_1)_\sigma$ and $\overline{V}(\kappa,\kappa_1)_\sigma$ corresponding to a given eigenvalue t_l, are the same. In particular, we have

$$\mathrm{Tr}_{V(\kappa,\kappa_1)_\sigma}(B_n^\sigma C_n) = \mathrm{Tr}_{\overline{V}(\kappa,\kappa_1)_\sigma}(B_n^\sigma C_n). \tag{2.5}$$

Next we expand $B_n^\sigma C_n$ in its semisimple and central parts,

$$B_n^\sigma C_n = \sum_{j=0}^{\infty} a_j Z_\sigma^j$$

in $\overline{\mathcal{A}}_\sigma$, with $a_j \in \overline{\mathcal{A}}_\sigma^{ss}$. Then (2.5) leads to the equality

$$\sum_{j=0}^{\infty} b_j \xi_\sigma^j = \sum_{j=0}^{\infty} b_j \overline{\xi}_\sigma^j \tag{2.6}$$

as formal power series in h, with

$$b_j = \mathrm{Tr}_{V(\kappa,\kappa_1)_\sigma}(a_j) \in \mathbb{C}[[h]].$$

Now the computations after Definition 1.5 show that

$$\mathrm{Ad}(g)\big(B_n^\sigma C_n|_{h=0}\big) = E_{n,n} - E_{n+1,n+1} - \sigma\, 1 = a_0|_{h=0} + a_1|_{h=0} Z'$$

with

$$\begin{aligned} a_0|_{h=0} = &-\frac{1}{n}E_{1,1} - \cdots - \frac{1}{n}E_{n-1,n-1} + \frac{n-1}{n}E_{n,n} \\ &+ \frac{1-n}{n}E_{n+1,n+1} + \frac{1}{n}E_{n+2,n+2} + \cdots + \frac{1}{n}E_{2n,2n} - \sigma\, 1 \end{aligned}$$

and $a_1|_{h=0} = -2$. Thus $b_1|_{h=0} \neq 0$ and $b_j|_{h=0} = 0$ for $j \geq 2$. Now we can expand (2.6) in powers of h, and compare coefficients of h^k on each side. An easy induction argument using $b_1|_{h=0} \neq 0$ and $b_j|_{h=0} = 0$ for $j \geq 2$ then leads to $\xi_\sigma = \overline{\xi}_\sigma$ (this may be viewed as a formal version of the inverse function theorem). $\square$

3. Zonal spherical functions

The purpose of this section is to establish Theorem 1.21 for $\vec{\kappa} = \vec{0} = (0,0,0)$. This is done by translating the results for zonal spherical functions on quantum Grassmannians proved in [19] and [2] to the present setting. In [19] and [2], extensive use is made of the L-operators of $U_q(\mathfrak{g})$. Roughly speaking, the L-operators $L^\pm$ for $U_q(\mathfrak{g})$ are the $2n \times 2n$ matrices with entries in $U_q(\mathfrak{g})$ obtained from the universal R-matrix of $U_q(\mathfrak{g})$ by applying the vector representation of $U_q(\mathfrak{g})$ to one of its components. We refer to Letzter [12, §6] and Noumi [18] for precise definitions, here we only recall those properties of the L-operators which are used in this paper. The matrix L^+ (resp. L^-) is upper (resp. lower) triangular. If $l_{ij}^\pm \in U_q(\mathfrak{g})$ is the (i,j)th matrix coefficient of $L^\pm$, then

$$\begin{aligned} l_{ii}^+ &= K_i, & l_{ii}^- &= K_i^{-1}, \\ l_{j,j+1}^+ &= (q-q^{-1})K_j y_j, & l_{j+1,j}^- &= (q^{-1}-q)x_j K_j^{-1} \end{aligned} \tag{3.1}$$

for $i=1,\ldots,2n$ and $j=1,\ldots,2n-1$. More generally, l_{ij}^{+} (resp. l_{ij}^{-}) is an analogue of the root vector of $\mathfrak{g}$ corresponding to the root $\epsilon_i-\epsilon_j$ (resp. $\epsilon_j-\epsilon_i$). Furthermore,

$$\Delta(l_{ij}^{\pm})=\sum_{k=1}^{2n} l_{ik}^{\pm}\otimes l_{kj}^{\pm},\qquad \epsilon(l_{ij}^{\pm})=\delta_{i,j},\qquad (l_{ij}^{\pm})^{*}=S(l_{ji}^{\mp})$$

and $S(L^{\pm})=(L^{\pm})^{-1}$, where the antipode is applied componentwise.

Define a scalar valued $2n\times 2n$ matrix J^{σ}, depending on an auxiliary parameter $\sigma\in\mathbb{R}$, by

$$J^{\sigma}=(1-q^{2\sigma})\sum_{k=1}^{n}E_{k,k}-q^{\sigma}\sum_{k=1}^{2n}E_{k,2n+1-k}.$$

The matrix J^{σ} is a solution of a reflection equation, see [19, Prop. 2.2] or [2, §6] for details. Observe furthermore that J^{σ} is invertible, with inverse

$$(J^{\sigma})^{-1}=(1-q^{-2\sigma})\sum_{k=1}^{n}E_{2n+1-k,2n+1-k}-q^{-\sigma}\sum_{k=1}^{2n}E_{k,2n+1-k}.$$

Definition 3.1 ([19], [2])**.** *Let $\mathfrak{k}_{\sigma}\subset U_q(\mathfrak{g})$ be the linear subspace spanned by the $4n^2$ matrix coefficients of $L^{+}J^{\sigma}-J^{\sigma}L^{-}$.*

Observe that $\mathfrak{k}_{\sigma}\subset U_q(\mathfrak{g})$ is a two-sided coideal, i.e. $\Delta(\mathfrak{k}_{\sigma})\subseteq\mathfrak{k}_{\sigma}\otimes U_q(\mathfrak{g})+U_q(\mathfrak{g})\otimes\mathfrak{k}_{\sigma}$ and $\epsilon(\mathfrak{k}_{\sigma})=0$. Note furthermore that $\mathfrak{k}_{\sigma}^{*}=S(\mathfrak{k}_{\sigma})$ since J^{σ} is symmetric, so $\mathfrak{k}_{\sigma}$ is *not* $*$-stable but $\mathfrak{k}_{\sigma}$ is ω-stable, where $\omega: U_q(\mathfrak{g})\to U_q(\mathfrak{g})$ is the involutive, anti-linear algebra isomorphism $\omega=*\circ S$. Note that the involution ω is related to the $*$-structure on $\mathbb{C}_q[G]$ by

$$f^{*}(X)=\overline{f(\omega(X))},\qquad f\in\mathbb{C}_q[G],\ X\in U_q(\mathfrak{g}). \tag{3.2}$$

It is convenient to write

$$\omega_{\delta}(X)=K^{-\delta}\omega(X)K^{\delta},\qquad X\in U_q(\mathfrak{g}). \tag{3.3}$$

Note that ω_{δ} is an anti-linear algebra involution of $U_q(\mathfrak{g})$, which acts on on the algebraic generators of $U_q(\mathfrak{g})$ by

$$\omega_{\delta}(K_i^{\pm 1})=K_i^{\mp 1},\qquad \omega_{\delta}(x_j)=-y_j,\qquad \omega_{\delta}(y_j)=-x_j$$

for $i=1,\ldots,2n$ and $j=1,\ldots,2n-1$.

For a subspace $U\subset U_q(\mathfrak{g})$, let $I(U)\subseteq U_q(\mathfrak{g})$ be the left ideal of $U_q(\mathfrak{g})$ generated by U and let $A(U)\subseteq U_q(\mathfrak{g})$ be the unital subalgebra generated by $I(U)$. Applied to $\mathfrak{k}_{\sigma}$ and $\mathfrak{k}_{\sigma}^{*}$, we thus obtain two unital subalgebras $A(\mathfrak{k}_{\sigma})$ and $A(\mathfrak{k}_{\sigma}^{*})$ of $U_q(\mathfrak{g})$. In the following lemma we link the algebras $A(\mathfrak{k}_{\sigma})$ and $A(\mathfrak{k}_{\sigma}^{*})$ to $\mathcal{A}_{\sigma}$.

Lemma 3.2. (i) *$\mathcal{A}_{\sigma}\cup\omega(\mathcal{A}_{\sigma})\subseteq A(\mathfrak{k}_{\sigma})$.*

(ii) *$\omega_{\delta}(\mathcal{A}_{\sigma})\subseteq A(\mathfrak{k}_{\sigma}^{*})$.*

Proof. **(i)** Note that $A(\mathfrak{k}_{\sigma})$ is ω-stable, because $\mathfrak{k}_{\sigma}$ is ω-stable. It thus suffices to prove that $\mathcal{A}_{\sigma}\subseteq A(\mathfrak{k}_{\sigma})$.

The left ideal $I(\mathfrak{k}_\sigma)$ contains the matrix coefficients of the $U_q(\mathfrak{g})$-valued matrix

$$(t_{i,j})_{i,j} = S(L^+)\big(L^+J^\sigma - J^\sigma L^-\big) = J^\sigma - S(L^+)J^\sigma L^-.$$

By direct computations one verifies that

$$\begin{aligned} t_{i,2n+1-i} &= -q^\sigma + q^\sigma C_i^{-1}, \qquad t_{j,2n-j} = q^\sigma(q^{-1}-q)B_j,\\ t_{n,n} &= q^\sigma(q^{-1}-q)B_n^\sigma + 1 - q^{2\sigma} \end{aligned}$$

for $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,2n-1\} \setminus \{n\}$. The left ideal $I(\mathfrak{k}_\sigma)$ also contains the matrix coefficients of

$$S(L^-)(J^\sigma)^{-1}(L^+J^\sigma - J^\sigma L^-)(J^\sigma)^{-1} = S(L^-)(J^\sigma)^{-1}L^+ - (J^\sigma)^{-1}.$$

Computing the coefficients of the antidiagonal shows that $q^{-\sigma} - q^{-\sigma}C_i \in I(\mathfrak{t}_\sigma)$ for $i \in \{1,\ldots,n\}$. Consequently, all algebraic generators of $\mathcal{A}_\sigma$ are contained in $A(\mathfrak{k}_\sigma)$, hence $\mathcal{A}_\sigma \subseteq A(\mathfrak{k}_\sigma)$.

(ii) Observe that

$$\begin{aligned} \omega_\delta(C_i^{\pm1}) &= C_i^{\mp1},\\ \omega_\delta(B_j) &= -q\big(K_{2n-j}y_{2n-j}K_j + K_{2n-j}K_{j+1}x_j\big),\\ \omega_\delta(B_n^\sigma) &= -\left(K_ny_nK_n + K_nK_{n+1}x_n + \left(\frac{q^\sigma - q^{-\sigma}}{q-q^{-1}}\right)K_n^2\right) \end{aligned} \tag{3.4}$$

for $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,2n-1\} \setminus \{n\}$ are algebraic generators of $\omega_\delta(\mathcal{A}_\sigma)$. Using similar arguments as in the proof of **(i)**, it can be shown that all these generators are contained in $A(\mathfrak{k}_\sigma^*) = A(S(\mathfrak{k}_\sigma))$. ☐

For a left $U_q(\mathfrak{g})$-module M and an unital subalgebra $A \subset U_q(\mathfrak{g})$ we define the subspace of A-invariant vectors in M by

$$M^A = \{m \in M \mid am = \epsilon(a)m \quad \forall\, a \in A\}.$$

Lemma 3.3. *For the $U_q(\mathfrak{g})$-subalgebras $A = \mathcal{A}_\sigma$, $A = \omega(\mathcal{A}_\sigma)$ and $A = \omega_\delta(\mathcal{A}_\sigma)$, we have*
(i) *$Dim\big(L_\lambda^A\big) = 0$ if $\lambda \in P_{2n}^+ \setminus (\Lambda_n^+)^\natural$.*
(ii) *$Dim\big(L_\lambda^A\big) \leq 1$ if $\lambda \in (\Lambda_n^+)^\natural$.*

Proof. The statement for $A = \mathcal{A}_\sigma$ follows from the results in §2.

For $\omega(\mathcal{A}_\sigma)$ and $\omega_\delta(\mathcal{A}_\sigma)$, we first remark that any simple, finite dimensional $U_q(\mathfrak{g})$-module L_λ ($\lambda \in P_{2n}^+$) is semisimple as module over the subalgebras $\omega(\mathcal{A}_\sigma)$ and $\omega_\delta(\mathcal{A}_\sigma)$. Clearly it suffices to prove this for $\omega_\delta(\mathcal{A}_\sigma)$. In this case we first note that

$$\omega_\delta(X^*) = \big(\omega_\delta(X)\big)^*, \qquad X \in U_q(\mathfrak{g}), \tag{3.5}$$

which follows easily from the fact that

$$S^2(X) = K^{-2\delta}XK^{2\delta}, \qquad X \in U_q(\mathfrak{g}). \tag{3.6}$$

We conclude that $\omega_\delta(\mathcal{A}_\sigma)$ is a $*$-subalgebra of $U_q(\mathfrak{g})$, hence L_λ is $\omega_\delta(\mathcal{A}_\sigma)$-semisimple.

Straightforward adjustments of the arguments in §2 now lead to the construction of expectation values and to the analogue of Proposition 2.2 for the algebras $\omega(\mathcal{A}_\sigma)$ and

$\omega_\delta(\mathcal{A}_\sigma)$. Since the elements C_i ($i = 1, \ldots, n$) are in $\omega(\mathcal{A}_\sigma)$ and $\omega_\delta(\mathcal{A}_\sigma)$, the lemma now follows for these two subalgebras in the same way as for $\mathcal{A}_\sigma$. $\square$

If the algebra $A \subseteq U_q(\mathfrak{g})$ is of the form $A = A(U)$ for some vector space $U \subset U_q(\mathfrak{g})$ and $\epsilon|_U \equiv 0$, then $M^{A(U)}$ consists of the vectors $m \in M$ which are annihilated by $u \in U$. In this situation, $M^{A(U)}$ is called the subspace of U-fixed vectors, cf. [19], [2]. In particular, this applies to $U = \mathfrak{k}_\sigma$ and $U = \mathfrak{k}_\sigma^*$. We now recall the following result, see [19, Thm. 2.6] and [2, Thm. 6.6].

Proposition 3.4. *For $U = \mathfrak{k}_\sigma$ and $U = \mathfrak{k}_\sigma^*$ we have*

$$\begin{aligned} Dim\big(L_\lambda^{A(U)}\big) &= 0 \quad \text{if} \quad \lambda \in P_{2n}^+ \setminus (\Lambda_n^+)^\natural, \\ Dim\big(L_\lambda^{A(U)}\big) &= 1 \quad \text{if} \quad \lambda \in (\Lambda_n^+)^\natural. \end{aligned}$$

Combined with Lemma 3.2 and Lemma 3.3, we obtain

Lemma 3.5. *For $\lambda \in P_{2n}^+$ we have*
(i) $L_\lambda^{\mathcal{A}_\sigma} = L_\lambda^{\omega(\mathcal{A}_\sigma)} = L_\lambda^{A(\mathfrak{k}_\sigma)}$.
(ii) $L_\lambda^{\omega_\delta(\mathcal{A}_\sigma)} = L_\lambda^{A(\mathfrak{k}_\sigma^*)}$.

Proof. By Lemma 3.3 and Proposition 3.4, all spaces in **(i)** and **(ii)** are $\{0\}$ when $\lambda \in P_{2n}^+ \setminus (\Lambda_n^+)^\natural$.

Let $\lambda \in (\Lambda_n^+)^\natural$. Lemma 3.2 implies $L_\lambda^{\mathcal{A}_\sigma} \supseteq L_\lambda^{A(\mathfrak{k}_\sigma)}$, $L_\lambda^{\omega(\mathcal{A}_\sigma)} \supseteq L_\lambda^{A(\mathfrak{k}_\sigma)}$ and $L_\lambda^{\omega_\delta(\mathcal{A}_\sigma)} \supseteq L_\lambda^{A(\mathfrak{k}_\sigma^*)}$. Since $\mathrm{Dim}(L_\lambda^A) = 1$ for $A = A(\mathfrak{k}_\sigma)$ and $A = A(\mathfrak{k}_\sigma^*)$ by Proposition 3.4, we conclude from Lemma 3.3 that $L_\lambda^{\mathcal{A}_\sigma} = L_\lambda^{A(\mathfrak{k}_\sigma)}$, $L_\lambda^{\omega(\mathcal{A}_\sigma)} = L_\lambda^{A(\mathfrak{k}_\sigma)}$ and $L_\lambda^{\omega_\delta(\mathcal{A}_\sigma)} = L_\lambda^{A(\mathfrak{k}_\sigma^*)}$. $\square$

The following result is a direct consequence of the definition of ω_δ and of Lemma 3.5.

Corollary 3.6. $L_\lambda^{A(\mathfrak{k}_\sigma^*)} = K^{-\delta} L_\lambda^{A(\mathfrak{k}_\sigma)}$ *for all $\lambda \in P_{2n}^+$.*

The main object of study in [19] and [2] is the space of $(\mathfrak{k}_\sigma, \mathfrak{k}_\tau)$-fixed regular functions on $U_q(\mathfrak{g})$, which is defined as follows.

Definition 3.7. *The space of $(\mathfrak{k}_\sigma, \mathfrak{k}_\tau)$-fixed functions on $U_q(\mathfrak{g})$ is defined by*

$$\mathcal{H}^{\sigma,\tau} = \{ f \in \mathbb{C}_q[G] \mid Y \cdot f = f \cdot Z = 0 \ \ \forall\, Y \in \mathfrak{k}_\sigma, \forall\, Z \in \mathfrak{k}_\tau \}.$$

Proposition 3.8. *The subspaces $\mathcal{H}^{\sigma,\tau}$ and $F_{\vec{0}}^{\sigma,\tau}$ of $\mathbb{C}_q[G]$ are related by*

$$\mathcal{H}^{\sigma,\tau} = F_{\vec{0}}^{\sigma,\tau} \cdot K^{-\delta}.$$

Proof. Any $f \in \mathcal{H}^{\sigma,\tau}$ can be written as a linear combination of functions

$$U_q(\mathfrak{g}) \ni X \mapsto \langle X v_\sigma, \widetilde{v}_\tau \rangle_\lambda$$

with $v_\sigma \in L_\lambda^{A(\mathfrak{k}_\sigma)}$, $\widetilde{v}_\tau \in L_\lambda^{A(\mathfrak{k}_\tau^*)}$ and $\lambda \in \big(\Lambda_n^+\big)^\natural$. On the other hand, any $g \in F_{\vec{0}}^{\sigma,\tau}$ can be written as a linear combination of functions

$$U_q(\mathfrak{g}) \ni X \mapsto \langle X v_\sigma, \widehat{v}_\tau \rangle_\lambda$$

with $v_\sigma \in L_\lambda^{\mathcal{A}_\sigma}$, $\widehat{v}_\tau \in L_\lambda^{\mathcal{A}_\tau}$ and $\lambda \in \left(\Lambda_n^+\right)^\natural$ since $\rho(0,0)_\sigma = \rho(0)_\sigma = \epsilon|_{\mathcal{A}_\sigma}$. The proposition follows now from the fact that $L_\lambda^{A(\mathfrak{k}_\sigma)} = L_\lambda^{\mathcal{A}_\sigma}$, $L_\lambda^{A(\mathfrak{k}_\tau^*)} = K^{-\delta} L_\lambda^{A(\mathfrak{k}_\tau)} = K^{-\delta} L_\lambda^{\mathcal{A}_\tau}$ and from the fact that $K^{-\delta}$ is $*$-selfadjoint. □

Remark 3.9. **(i)** The fact that $\mathfrak{k}_\sigma$ and $\mathfrak{k}_\tau$ are ω-stable two-sided coideals implies that $\mathcal{H}^{\sigma,\tau} \subseteq \mathbb{C}_q[G]$ is a unital $*$-subalgebra, cf. [19] and [2]. Furthermore, $K^{-\delta} \in U_q(\mathfrak{g})$ is a group-like element, so the previous proposition shows that $F_{\vec{0}}^{\sigma,\tau} \subseteq \mathbb{C}_q[G]$ is a unital subalgebra (but it is not $*$-stable!).

(ii) Observe that $\epsilon(\omega_\delta(X)) = \overline{\epsilon(X)}$ for $X \in U_q(\mathfrak{g})$. Combined with Lemma 3.5, the proof of Proposition 3.8 and (3.5), this leads to the alternative description

$$\mathcal{H}^{\sigma,\tau} = \{f \in \mathbb{C}_q[G] \,|\, f(\omega_\delta(b)Xa) = \epsilon(a)\overline{\epsilon(b)}f(X), \qquad \forall\, X \in U_q(\mathfrak{g}),\ \forall\, a \in \mathcal{A}_\sigma,\ \forall\, b \in \mathcal{A}_\tau\}$$

for the space of $(\mathfrak{k}_\sigma, \mathfrak{k}_\tau)$-fixed regular functions on $U_q(\mathfrak{g})$.

Recall that the restriction map $|_T : F_{\vec{0}}^{\sigma,\tau} \to \mathbb{C}[u^{\pm 1}]$ involved a δ-shift: $f|_T(q^\lambda) = f(K^{\lambda-\delta})$ for all $\lambda \in P_{2n}$. When dealing with the $*$-subalgebra $\mathcal{H}^{\sigma,\tau}$, it is convenient to use the restriction map without δ-shift,

$$\mathrm{Res}_T : \mathcal{H}^{\sigma,\tau} \to \mathbb{C}[u^{\pm 1}], \qquad \mathrm{Res}_T(f)(q^\lambda) = f(K^\lambda) \quad \forall\, \lambda \in P_{2n},$$

since then we have

$$\mathrm{Res}_T(f \cdot K^{-\delta}) = f|_T, \qquad \forall\, f \in F_{\vec{0}}^{\sigma,\tau},$$

cf. Proposition 3.8. Note that Res_T is a $*$-algebra homomorphism, with the $*$-structure on $\mathbb{C}[u^{\pm 1}]$ defined by

$$p^* = \sum_{\mu \in \Lambda_n} \overline{c_\mu} u^{-\mu} \quad \text{if} \quad p = \sum_{\mu \in \Lambda_n} c_\mu u^\mu \in \mathbb{C}[u^{\pm 1}]. \tag{3.7}$$

We are now in a position to translate the main result from [19] to our present setup.

Proposition 3.10. *Theorem 1.21 is valid for $\vec{\kappa} = \vec{0}$.*

Proof. Observe that $\delta(0,0) = 0$, hence we can take the unit $1 \in \mathbb{C}_q[G]$ as nonzero element in $F_{\vec{0}}^{\sigma,\tau}(0)$. Clearly $1|_T$ is the unit of $\mathbb{C}[u^{\pm 1}]$. This gives Theorem 1.21**(ii)** for $\vec{\kappa} = \vec{0}$.

The generalized Chevalley restriction theorem (Theorem 1.21**(i)**) for $\vec{\kappa} = \vec{0}$ reduces to the statement that $|_T : F_{\vec{0}}^{\sigma,\tau} \to \mathbb{C}[u^{\pm 1}]^W$ is a linear isomorphism. In view of Proposition 3.8 and the δ-shift in the definition of the restriction map $|_T$, this is equivalent to the statement that $\mathrm{Res}_T : \mathcal{H}^{\sigma,\tau} \to \mathbb{C}[u^{\pm 1}]^W$ is a linear isomorphism, which follows from [19, Thm. 3.2], see also [2, Thm. 7.5]. For a direct proof of this result for an arbitrary quantum symmetric pair, see [15].

Let $0 \neq f_\mu \in F_{\vec{0}}^{\sigma,\tau}(\mu)$ be an elementary spherical function of degree $\mu \in \Lambda_n^+$, then $f_\mu \cdot K^{-\delta} \in \mathcal{H}^{\sigma,\tau} \cap W(\mu^\natural)$. Consequently, [19, Thm. 3.4] shows that the restriction $f_\mu|_T = \mathrm{Res}_T(f_\mu \cdot K^{-\delta}) \in \mathbb{C}[u^{\pm 1}]^W$ is a nonzero scalar multiple of the Macdonald-Koornwinder polynomial

$$P_\mu^{\sigma,\tau}(u) = P_\mu(u; -q^{\sigma+\tau+1}, -q^{-\sigma-\tau+1}, q^{\sigma-\tau+1}, q^{-\sigma+\tau+1}; q^2, q^2), \tag{3.8}$$

(see also [2, Thm. 7.5]). This proves Theorem 1.21**(iii)** for $\vec{\kappa}=\vec{0}$. $\square$

4. The generalized Chevalley theorem

In this section we prove the generalized Chevalley restriction theorem (see Theorem 1.21**(i)**).

To simplify notations, we write $[\cdot,\cdot]_\mu$ for the scalar product $\langle\cdot,\cdot\rangle_{(\mu+\delta(\kappa,\kappa_1))^\natural}$ on the $U_q(\mathfrak{g})$-module $L_{(\mu+\delta(\kappa,\kappa_1))^\natural}$ ($\mu\in\Lambda_n^+$).

Lemma 4.1. *Let $\mu\in\Lambda_n^+$ and identify $V(\kappa,\kappa_1)_\tau$ with its unique copy in the semisimple $\mathcal{A}_\tau$-module $L^\tau_{(\mu+\delta(\kappa,\kappa_1))^\natural}$ and $V(\kappa_2)_\sigma$ with its unique copy in the semisimple $\mathcal{A}_\sigma$-module $L^\sigma_{(\mu+\delta(\kappa,\kappa_1))^\natural}$. Let $f_\mu\in F^{\sigma,\tau}_{\vec{\kappa}}(\mu)$ be an elementary vector valued spherical function of degree μ. Then*

$$f_\mu|_T(q^\lambda)=[K^{\lambda-\delta}v,w]_\mu,\qquad \forall\,\lambda\in P_{2n}$$

for suitably normalized vectors $0\neq v\in V(\kappa_2)_\sigma$ and $0\neq w\in\widetilde{V}(\kappa,\kappa_1)_\tau$. Furthermore,

$$f_\mu|_T=\sum_{-\mu-\delta(\kappa,\kappa_1)\leq\nu\leq\mu+\delta(\kappa,\kappa_1)}c_\nu u^\nu\in\mathbb{C}[u^{\pm1}]$$

for certain constants $c_\nu\in\mathbb{C}$ ($\nu\in\Lambda_n$), and $c_{\pm\mu\pm\delta(\kappa,\kappa_1)}\neq0$.

Proof. Choose an orthonormal basis $\{b_1,\ldots,b_m\}$ of $V(\kappa,\kappa_1)_\tau$ with respect to the restriction of $[\cdot,\cdot]_\mu$ to $V(\kappa,\kappa_1)_\tau$ such that $b_1\in\widetilde{V}(\kappa,\kappa_1)_\tau$. In particular, b_1 is a nonzero constant multiple of w. Then $f_\mu\in F^{\sigma,\tau}_{\vec{\kappa}}(\mu)$ can be realized as

$$f_\mu(X)=\sum_{i=1}^m[Xv,b_i]_\mu\,b_i,\qquad X\in U_q(\mathfrak{g})$$

for some $0\neq v\in V(\kappa_2)_\sigma$. We have $f_\mu(X)\in\widetilde{V}(\kappa,\kappa_1)_\tau$ for $X\in U^0$, see (the proof of) Lemma 1.20. Using the identification $\widetilde{V}(\kappa,\kappa_1)_\tau\simeq\mathbb{C}$ of vectorspaces given by the assignment $b_1\mapsto1$, we obtain

$$f_\mu|_T(q^\lambda)=f_\mu(K^{\lambda-\delta})=[K^{\lambda-\delta}v,b_1]_\mu,\qquad\forall\,\lambda\in P_{2n}.$$

Now both vectors v and b_1 satisfy $C_iv=v$ and $C_ib_1=b_1$ for $i=1,\ldots,n$, hence $v[\lambda]=0$ and $b_1[\lambda]=0$ for $\lambda\in P_{2n}$ unless $\lambda\in\big(\Lambda_n\big)^\natural$. Since $\natural:\Lambda_n\to\big(\Lambda_n\big)^\natural\subset P_{2n}$ is a bijection of partially ordered sets and $(-\nu)^\natural=w_0(\nu^\natural)$ for $\nu\in\Lambda_n$, we conclude that the weight decompositions of v and b_1 are of the form

$$v=\sum_{-\mu-\delta(\kappa,\kappa_1)\leq\nu\leq\mu+\delta(\kappa,\kappa_1)}v[\nu^\natural],\qquad b_1=\sum_{-\mu-\delta(\kappa,\kappa_1)\leq\nu\leq\mu+\delta(\kappa,\kappa_1)}b_1[\nu^\natural],$$

with $v[\nu^\natural]$ and $b_1[\nu^\natural]$ the components of the vectors v and b_1 in the weight space of the $U_q(\mathfrak{g})$-module $L_{(\mu+\delta(\kappa,\kappa_1))^\natural}$ of weight $\nu^\natural$.

A simple application of Lemma 2.5 shows that $v[(\pm\mu\pm\delta(\kappa,\kappa_1))^\natural]\neq 0$. Denote $\widetilde{L}^\tau_{(\mu+\delta(\kappa,\kappa_1))^\natural}$ for the vectors $v\in L^\tau_{(\mu+\delta(\kappa,\kappa_1))^\natural}$ satisfying $C_iv=v$ for $i=1,\ldots,n$. This subspace contains the one dimensional weight spaces $L_{(\mu+\delta(\kappa,\kappa_1))^\natural}[(\pm\mu\pm\delta(\kappa,\kappa_1))^\natural]$ and $\widetilde{L}^\tau_{(\mu+\delta(\kappa,\kappa_1))^\natural}\cap\widetilde{V}(\kappa,\kappa_1)_\tau=\mathbb{C}\{b_1\}$. Lemma 2.5 implies $b_1[(\pm\mu\pm\delta(\kappa,\kappa_1))^\natural]\neq 0$. Hence

$$\begin{aligned}f_\mu|_T&=\sum_{-\mu-\delta(\kappa,\kappa_1)\leq\nu\leq\mu+\delta(\kappa,\kappa_1)}[K^{-\delta}v[\nu^\natural],b_1[\nu^\natural]]_\mu\,u^\nu\\&=\sum_{-\mu-\delta(\kappa,\kappa_1)\leq\nu\leq\mu+\delta(\kappa,\kappa_1)}c_\nu u^\nu\end{aligned}$$

with $c_\nu\in\mathbb{C}$ satisfying the conditions as stated in the lemma. □

A more detailed analysis of the elementary vector valued spherical functions requires the right coideal algebra structure of $\mathcal{A}_\sigma$. Recall that a right $U_q(\mathfrak{g})$-comodule (M,δ_M) is a vectorspace M together with a linear map $\delta_M:M\to M\otimes U_q(\mathfrak{g})$ satisfying

$$\big(\delta_M\otimes\mathrm{Id}_{U_q(\mathfrak{g})}\big)\circ\delta_M=\big(\mathrm{Id}_M\otimes\Delta\big)\circ\delta_M,\qquad\big(\mathrm{Id}_M\otimes\epsilon\big)\circ\delta_M=\mathrm{Id}_M.$$

If M is a unital algebra and δ_M is a (unit preserving) algebra homomorphism, then (M,δ_M) is called a right $U_q(\mathfrak{g})$-comodule algebra. In particular, a unital subalgebra $A\subseteq U_q(\mathfrak{g})$ satisfying $\Delta(A)\subseteq A\otimes U_q(\mathfrak{g})$ gives rise to the right comodule algebra $(A,\Delta|_A)$. In this case, A is called a right coideal subalgebra of $U_q(\mathfrak{g})$.

By computing the action of the comultiplication Δ on the algebraic generators of $\mathcal{A}_\sigma$, one verifies that $\mathcal{A}_\sigma\subseteq U_q(\mathfrak{g})$ is a right $U_q(\mathfrak{g})$-coideal algebra. Using the isomorphism $\pi_\sigma:\mathcal{A}\to\mathcal{A}_\sigma$ of algebras, see Proposition 1.4, $\mathcal{A}$ inherets a unique right $U_q(\mathfrak{g})$-comodule structure such that $\pi_\sigma:\mathcal{A}\to\mathcal{A}_\sigma$ is an isomorphism of right $U_q(\mathfrak{g})$-comodule algebras. The right comodule map for $\mathcal{A}$ turns out to be independent of σ, and is given as follows, cf., e.g., [14].

Proposition 4.2. *The algebra $\mathcal{A}$ has the structure of a right $U_q(\mathfrak{g})$-comodule algebra, with comodule map $\delta_{\mathcal{A}}$ given explicitly by*

$$\begin{aligned}\delta_{\mathcal{A}}(\gamma_i^{\pm1})&=\gamma_i^{\pm1}\otimes K_i^{\pm1}K_{2n+1-i}^{\pm1},\\ \delta_{\mathcal{A}}(\beta_j)&=\beta_j\otimes K_j^{-1}K_{2n-j}^{-1}+\gamma_{j+1}^{-1}\otimes y_jK_{j+1}^{-1}K_{2n-j}^{-1}+\gamma_j^{-1}\otimes K_j^{-1}x_{2n-j}K_{2n-j}^{-1}\end{aligned}$$

for $i=1,\ldots,2n$ and $j=1,\ldots,2n-1$. The map $\pi_\sigma:\mathcal{A}\to\mathcal{A}_\sigma$ is an isomorphism of right $U_q(\mathfrak{g})$-comodule algebras.

Finite dimensional $\mathcal{A}_\sigma$-modules do not form a tensor category, since $\mathcal{A}_\sigma\subset U_q(\mathfrak{g})$ is not a coalgebra. The right coideal algebra structure of $\mathcal{A}_\sigma$ implies though that the tensor product $M\otimes N$ of an $\mathcal{A}_\sigma$-module M and an $U_q(\mathfrak{g})$-module N is an $\mathcal{A}_\sigma$-module by the usual formula

$$a\cdot(m\otimes n)=\sum a_1\cdot m\otimes a_2\cdot n,\qquad m\in M,\ n\in N,$$

where (recall) $\Delta(a)=\sum a_1\otimes a_2$ for $a\in\mathcal{A}_\sigma$ with the a_1's from $\mathcal{A}_\sigma$. In the special case that M is the one-dimensional $\mathcal{A}_\sigma$-module $V(k)_\sigma$ ($k\in\mathbb{Z}$), we can naturally relate the $\mathcal{A}_\sigma$-module $V(k)_\sigma\otimes N$ to the $\mathcal{A}_{\sigma+2k}$-module $N^{\sigma+2k}$ in the following manner.

Let $\chi_k^\sigma : \mathcal{A}_\sigma \to \mathbb{C}$ be the character of $V(k)_\sigma$. The $\mathcal{A}_\sigma$-action on $V(k)_\sigma \otimes N$ can then be rewritten as

$$a \cdot (m \otimes n) = m \otimes \zeta_k^\sigma(a) \cdot n, \qquad a \in \mathcal{A}_\sigma,\ m \in V(k)_\sigma,\ n \in N,$$

with $\zeta_k^\sigma : \mathcal{A}_\sigma \to U_q(\mathfrak{g})$ the unital algebra homomorphism

$$\zeta_k^\sigma(a) = \sum \chi_k^\sigma(a_1) a_2, \qquad a \in \mathcal{A}_\sigma. \tag{4.1}$$

Lemma 4.3. *The map ζ_k^σ defines an algebra isomorphism $\zeta_k^\sigma : \mathcal{A}_\sigma \to \mathcal{A}_{\sigma+2k}$, with inverse $\zeta_{-k}^{\sigma+2k} : \mathcal{A}_{\sigma+2k} \to \mathcal{A}_\sigma$. Furthermore, $\zeta_k^\sigma = \pi_{\sigma+2k} \circ \pi_\sigma^{-1}|_{\mathcal{A}_\sigma}$ (see Proposition 1.4 for the definition of π_σ), which means that*

$$\zeta_k^\sigma(C_i) = C_i, \qquad \zeta_k^\sigma(B_j) = B_j, \qquad \zeta_k^\sigma(B_n^\sigma) = B_n^{\sigma+2k} \tag{4.2}$$

for $i \in \{1, \ldots, 2n\}$ and $j \in \{1, \ldots, 2n-1\} \setminus \{n\}$.

Proof. By Lemma 1.13 the values of χ_k^σ on the algebraic generators of $\mathcal{A}_\sigma$ are given by

$$\chi_k^\sigma(C_i) = 1, \qquad \chi_k^\sigma(B_j) = 0, \qquad \chi_k^\sigma(B_n^\sigma) = \vartheta_0(q^{\sigma+2k}) \tag{4.3}$$

for $i \in \{1, \ldots, 2n\}$ and $j \in \{1, \ldots, 2n-1\} \setminus \{n\}$, with $\vartheta_l(s)$ ($l \in \mathbb{Z}$) defined by

$$\vartheta_l(s) = \frac{s^{-1} - sq^{-2l}}{q - q^{-1}}.$$

Combined with Proposition 4.2 one now easily derives (4.2). The remaining statements follow now immediately. $\square$

Corollary 4.4. *Let N be a finite dimensional $U_q(\mathfrak{g})$-module and suppose that $N' \subset N^{\sigma+2k}$ is an $\mathcal{A}_{\sigma+2k}$-submodule.*

(i) *$V(k)_\sigma \otimes N'$ is an $\mathcal{A}_\sigma$-submodule of $V(k)_\sigma \otimes N$.*

(ii) *If $N' \simeq V(\kappa, \kappa_1)_{\sigma+2k}$, then $V(k)_\sigma \otimes N' \simeq V(\kappa, \kappa_1 + k)_\sigma$ as $\mathcal{A}_\sigma$-modules.*

Proof. **(i)** follows from Lemma 4.3, and **(ii)** follows from Lemma 4.3 and Lemma 1.13. $\square$

As a first application of these tensor product constructions we derive (a refinement of) the generalized Chevalley restriction theorem (see Theorem 1.21**(i)**). Before stating the result, we first recall some basic facts on tensor products of finite dimensional $U_q(\mathfrak{g})$-modules which we need in the proof. For any $\lambda, \mu \in P_{2n}^+$, the irreducible decomposition of the finite dimensional $U_q(\mathfrak{g})$-module $L_\lambda \otimes L_\mu$ is of the form

$$L_\lambda \otimes L_\mu = L_{\lambda+\mu} \oplus \bigoplus_{\substack{\nu \in P_{2n}^+ \\ \nu \prec \lambda+\mu}} L_\nu^{\oplus d_\nu^{\lambda,\mu}} \tag{4.4}$$

for certain multiplicities $d_\nu^{\lambda,\mu} \in \mathbb{Z}_{\geq 0}$. The copy $L_{\lambda+\mu}$ has $v_\lambda \otimes v_\mu$ as highest weight vector. We write $\mathrm{pr}_{\lambda,\mu} : L_\lambda \otimes L_\mu \to L_{\lambda+\mu}$ for the projection along the direct sum decomposition (4.4), and $\mathrm{pr}_\mu = \mathrm{pr}_{\mu^\natural, \delta(\kappa,\kappa_1)^\natural}$ for $\mu \in \Lambda_n^+$.

Proposition 4.5. *Let $\mu \in \Lambda_n^+$ and choose elementary vector valued spherical functions $f_\mu \in F_{\vec{\kappa}}^{\sigma,\tau}(\mu)$ and $f_0 \in F_{\vec{\kappa}}^{\sigma,\tau}(0)$. Then $f_\mu|_T$ is divisible by $f_0|_T$ in $\mathbb{C}[u^{\pm 1}]$, and*

$$\frac{f_\mu|_T}{f_0|_T} = \sum_{\substack{\nu \in \Lambda_n^+ \\ \nu \leq \mu}} c_\nu m_\nu \in \mathbb{C}[u^{\pm 1}]^W \tag{4.5}$$

for some constants $c_\nu \in \mathbb{C}$ with $c_\mu \neq 0$.

Proof. The proof is by induction on $\mu \in \Lambda_n^+$ along the dominance order $\leq$. For $\mu = 0$ the proposition is trivial. We identify $V(0)_\sigma$ with its unique copy in $L_{\mu^\natural}^\sigma$ and $V(\kappa_2)_\sigma$ with its unique copy in $L_{\delta(\kappa,\kappa_1)^\natural}^\sigma$, and we choose nonzero vectors

$$w_\sigma \in V(0)_\sigma \subset L_{\mu^\natural}^\sigma, \qquad u_\sigma \in V(\kappa_2)_\sigma \subset L_{\delta(\kappa,\kappa_1)^\natural}^\sigma.$$

We furthermore fix nonzero intertwiners

$$\phi_\mu \in \mathrm{Hom}_{\mathcal{A}_\tau}(L_{\mu^\natural}^\tau, V(0)_\tau), \qquad \phi_0 \in \mathrm{Hom}_{\mathcal{A}_\tau}(L_{\delta(\kappa,\kappa_1)^\natural}^\tau, V(\kappa,\kappa_1)_\tau)$$

and we identify $V(0)_\tau \simeq \mathbb{C}$ as vector spaces. Then

$$g_\mu(X) = \phi_\mu(Xw_\sigma), \qquad f_0(X) = \phi_0(Xu_\sigma)$$

for $X \in U_q(\mathfrak{g})$ define elementary vector valued spherical functions $g_\mu \in F_{\vec{0}}^{\sigma,\tau}(\mu)$ and $f_0 \in F_{\vec{\kappa}}^{\sigma,\tau}(0)$. Furthermore, $g_\mu|_T$ is of the form

$$g_\mu|_T = \sum_{\substack{\nu \in \Lambda_n^+ \\ \nu \leq \mu}} d_\nu m_\nu \tag{4.6}$$

for some constants $d_\nu \in \mathbb{C}$ with $d_\mu \neq 0$ in view of Proposition 3.10 (in fact, $g_\mu|_T$ is a nonzero constant multiple of a Macdonald-Koornwinder polynomial of degree μ).

We now consider the linear map $f : U_q(\mathfrak{g}) \to V(\kappa,\kappa_1)_\tau$ defined by

$$f(X) = \phi\big(X(w_\sigma \otimes u_\sigma)\big), \qquad X \in U_q(\mathfrak{g}),$$

with $\phi : L_{\mu^\natural} \otimes L_{\delta(\kappa,\kappa_1)^\natural} \to V(\kappa,\kappa_1)_\tau$ the linear map defined by

$$\phi(u \otimes v) = \phi_\mu(u)\phi_0(v), \qquad u \in L_{\mu^\natural},\ v \in L_{\delta(\kappa,\kappa_1)^\natural}.$$

Observe that $\mathbb{C}\{w_\sigma \otimes u_\sigma\}$ is an $\mathcal{A}_\sigma$-submodule of $\big(L_{\mu^\natural} \otimes L_{\delta(\kappa,\kappa_1)^\natural}\big)^\sigma$ which is isomorphic to $V(\kappa_2)_\sigma$ by Corollary 4.4. Furthermore,

$$\phi \in \mathrm{Hom}_{\mathcal{A}_\tau}\big((L_{\mu^\natural} \otimes L_{\delta(\kappa,\kappa_1)^\natural})^\tau, V(\kappa,\kappa_1)_\tau\big),$$

since

$$\begin{aligned}\phi(a(v \otimes w)) &= \sum \phi_\mu(a_1 v)\phi_0(a_2 w)\\ &= \phi_\mu(v)\phi_0(\zeta_0^\tau(a)w)\\ &= \rho(\kappa,\kappa_1)_\tau(a)\phi(v \otimes w)\end{aligned}$$

for $a \in \mathcal{A}_\tau$, $v \in L_{\mu^\natural}$ and $w \in L_{\delta(\kappa,\kappa_1)^\natural}$, where, besides the intertwining properties of ϕ_μ and ϕ_0, we have used that $\zeta_0^\tau = \mathrm{Id}_{\mathcal{A}_\tau}$. Hence we conclude that $f \in F_{\vec{\kappa}}^{\sigma,\tau}$.

By (4.4) and Corollary 1.18 we have an expansion

$$f = \sum_{\substack{\nu' \in \Lambda_n^+ \\ \nu' \leq \mu}} f_{\nu'} \tag{4.7}$$

with $f_{\nu'} \in F_{\vec{\kappa}}^{\sigma,\tau}(\nu')$ and $f_\mu(\cdot) = \phi(\cdot\,\mathrm{pr}_\mu(w_\sigma \otimes u_\sigma))$. Lemma 2.5 implies that $\mathrm{pr}_\mu(w_\sigma \otimes u_\sigma)$ spans the unique copy of $V(\kappa_2)_\sigma$ in $L^\sigma_{(\mu+\delta(\kappa,\kappa_1))^\natural}$. Furthermore, $\phi \circ \mathrm{pr}_\mu \neq 0$ since

$$\phi(v_{\mu^\natural} \otimes v_{\delta(\kappa,\kappa_1)^\natural}) = \phi_\mu(v_{\mu^\natural})\phi_0(v_{\delta(\kappa,\kappa_1)^\natural}) \neq 0 \tag{4.8}$$

by Proposition 2.2. Consequently, f_μ is nonzero.

Up to a nonzero multiplicative constant, f_μ is the unique elementary vector valued spherical function of degree μ. It thus suffices to prove the induction step for f_μ. By the construction of the vector valued spherical function f, we have

$$f|_T = (g_\mu|_T)(f_0|_T) \tag{4.9}$$

in $\mathbb{C}[u^{\pm 1}]$. Substituting the expansions (4.6) and (4.7) of $g_\mu|_T$ and f respectively in (4.9), gives

$$\sum_{\substack{\nu \in \Lambda_n^+ \\ \nu \leq \mu}} d_\nu\, f_0|_T m_\nu = f|_T = f_\mu|_T + \sum_{\substack{\nu' \in \Lambda_n^+ \\ \nu' < \mu}} f_{\nu'}|_T.$$

By the induction hypothesis applied to the $f_{\nu'}|_T$'s ($\nu' < \mu$), we conclude that (4.5) is valid for some $c_\nu \in \mathbb{C}$ and that $c_\mu = d_\mu \neq 0$. This completes the proof of the induction step. □

As an immediate corollary of Proposition 4.5, we obtain the generalized Chevalley restriction theorem (see Theorem 1.21**(i)**):

Corollary 4.6. *Let $f_0 \in F_{\vec{\kappa}}^{\sigma,\tau}(0)$ be an elementary vector valued spherical function of degree zero. The restriction map $|_T$ induces a linear bijection $|_T : F_{\vec{\kappa}}^{\sigma,\tau} \to f_0|_T\, \mathbb{C}[u^{\pm 1}]^W$.*

5. The ground state

We define a ground state f_0 (with respect to $(\vec{\kappa}, \sigma, \tau)$) to be an elementary vector valued spherical function $f_0 \in F_{\vec{\kappa}}^{\sigma,\tau}(0)$ of degree zero. Note that a ground state f_0 is unique up to nonzero scalar multiples. In this section we compute the radial part $f_0|_T$ of a ground state f_0 explicitly. In the first subsection we show that it suffices to compute the ground state for $\vec{\kappa} = (0, 0, \kappa)$ and $\vec{\kappa} = (\kappa_1, \kappa_2, 0)$ seperately. The second and third subsections are devoted to the explicit computation of the ground state for these two special cases.

5.1. **Splitting of the ground state.** For the computation of the radial part of the ground state it is sufficient to compute the radial part for the special cases $\kappa_1 = \kappa_2 = 0$ and $\kappa = 0$ in view of the following lemma.

Lemma 5.1. *Let $f \in F^{\sigma,\tau}_{(\kappa_1,\kappa_2,0)}(0)$ and $f' \in F^{\sigma+2\kappa_2,\tau+2\kappa_1}_{(0,0,\kappa)}(0)$ be ground states with respect to $(\kappa_1, \kappa_2, 0, \sigma, \tau)$ and $(0, 0, \kappa, \sigma + 2\kappa_2, \tau + 2\kappa_1)$, respectively. Then $f|_T f'|_T = f_0|_T$ in $\mathbb{C}[u^{\pm 1}]$ with $f_0 \in F_{\vec{\kappa}}^{\sigma,\tau}(0)$ a ground state with respect to $(\vec{\kappa}, \sigma, \tau)$.*

Proof. We proceed as in the first part of the proof of Proposition 4.5. We identify $V(\kappa_2)_\sigma$ with its unique copy in $L^{\sigma}_{\delta(0,\kappa_1)^\natural}$ and $V(0)_{\sigma+2\kappa_2}$ with its unique copy in $L^{\sigma+2\kappa_2}_{\delta(\kappa,0)^\natural}$, and we fix nonzero vectors $u\in V(\kappa_2)_\sigma$ and $v\in V(0)_{\sigma+2\kappa_2}$. We furthermore fix nonzero intertwiners

$$\varphi\in \mathrm{Hom}_{\mathcal{A}_\tau}\bigl(L^{\tau}_{\delta(0,\kappa_1)^\natural},V(\kappa_1)_\tau\bigr),\qquad \phi\in \mathrm{Hom}_{\mathcal{A}_{\tau+2\kappa_1}}\bigl(L^{\tau+2\kappa_1}_{\delta(\kappa,0)^\natural},V(\kappa,0)_{\tau+2\kappa_1}\bigr),$$

then the formulas

$$f(X)=\varphi(Xu),\qquad f'(X)=\phi(Xv)$$

for $X\in U_q(\mathfrak{g})$ define ground states $f\in F^{\sigma,\tau}_{(\kappa_1,\kappa_2,0)}(0)$ and $f'\in F^{\sigma+2\kappa_2,\tau+2\kappa_1}_{(0,0,\kappa)}(0)$. We now consider the function $f_0(X)=\psi(X(u\otimes v))$ for $X\in U_q(\mathfrak{g})$, with

$$\psi: L_{\delta(0,\kappa_1)^\natural}\otimes L_{\delta(\kappa,0)^\natural}\to V(\kappa_1)_\tau\otimes V(\kappa,0)_{\tau+2\kappa_1}\simeq V(\kappa,\kappa_1)_\tau \tag{5.1}$$

defined by $\psi(w\otimes w')=\varphi(w)\otimes\phi(w')$ for $w\in L_{\delta(0,\kappa_1)^\natural}$ and $w'\in L_{\delta(\kappa,0)^\natural}$ (see Corollary 4.4 for the isomorphism $V(\kappa_1)_\tau\otimes V(\kappa,0)_{\tau+2\kappa_1}\simeq V(\kappa,\kappa_1)_\tau$ of $\mathcal{A}_\tau$-modules in (5.1)). Similarly as in the proof of Proposition 4.5 we have

$$f_0|_T=(f|_T)(f'|_T)$$

in $\mathbb{C}[u^{\pm1}]$ and $0\neq f_0\in F^{\sigma,\tau}_{\vec{\kappa}}$. In the expansion of f_0 in elementary vector valued spherical functions, there is only a contribution of degree zero in view of (4.4) and Corollary 1.18, hence we conclude that $f_0\in F^{\sigma,\tau}_{(\kappa_1,\kappa_2,\kappa)}(0)$ is the ground state with respect to $(\vec{\kappa},\sigma,\tau)$. ☐

5.2. **The ground state for $\kappa_1=\kappa_2=0$.** We compute the radial part $f_0|_T\in\mathbb{C}[u^{\pm1}]$ of the ground state $f_0\in F^{\sigma,\tau}_{(0,0,\kappa)}(0)$ by relating $f_0|_T$ to the ground state for vector-valued $U_q(\mathfrak{gl}(n))$-characters, which in turn was computed explicitly by Etingof and Kirillov [3].

We use the notations of §1. Denote (ρ_κ,V_κ) for the finite dimensional, irreducible $U_q(\mathfrak{gl}(n))$-representation $L_{(n-1)\kappa,(-\kappa)^{n-1}}$. We view an element $f\in\mathbb{C}_q[\mathrm{GL}(n;\mathbb{C})]^{\otimes2}\otimes V$ as a linear map $f:U_q(\mathfrak{gl}(n))^{\otimes2}\to V$.

Definition 5.2. *Let $\widetilde{F}_\kappa$ be the space of functions $f\in\mathbb{C}_q[GL(n;\mathbb{C})]^{\otimes2}\otimes V_\kappa$ satisfying*

$$\begin{aligned} f(X\Delta^{op}(a))&=\epsilon(a)f(X), &&\forall\, a\in U_q(\mathfrak{gl}(n)),\\ f(\Delta^{op}(b)X)&=\rho_\kappa(b)f(X), &&\forall\, b\in U_q(\mathfrak{gl}(n))\end{aligned}$$

for all $X\in U_q(\mathfrak{gl}(n))^{\otimes2}$.

Recall from Lemma 1.14 that the isomorphism $U_q(\mathfrak{gl}(n))\simeq\mathcal{E}$ of Lemma 1.8 implies that the restriction of $V(\kappa,\kappa_1)_\tau$ to $\mathcal{E}$ is isomorphic to V_κ as $U_q(\mathfrak{gl}(n))$-modules. In the following lemma we furthermore use the natural identification $U_q(\mathfrak{gl}(n))^{\otimes2}\simeq U_q(\mathfrak{k})\subset U_q(\mathfrak{g})$, cf. §1.3.

Lemma 5.3. *The assignment*

$$f\mapsto f\circ(Id\otimes\psi),$$

with ψ defined by (1.14)*, defines a linear map $F^{\sigma,\tau}_{\vec{\kappa}}\to\widetilde{F}_\kappa$.*

Proof. With the identifications discussed above, the lemma follows immediately from the explicit form of the isomorphism $U_q(\mathfrak{gl}(n))\simeq\mathcal{E}$, see Lemma 1.8. ☐

For a left $U_q(\mathfrak{gl}(n))$-module V, we consider its linear dual V^* as $U_q(\mathfrak{gl}(n))$-module by

$$(X\phi)(v) = \phi(S^{-1}(X)v), \qquad X \in U_q(\mathfrak{gl}(n)),\, \phi \in V^*,\, v \in V.$$

Let $(V \otimes V^*)^{op}$ be the $U_q(\mathfrak{gl}(n))$-module with representation space $V \otimes V^*$ and $U_q(\mathfrak{gl}(n))$-action

$$X(v \otimes \phi) = \sum X_2 v \otimes X_1 \phi, \qquad X \in U_q(\mathfrak{gl}(n)),\ v \in V,\ \phi \in V^*.$$

Lemma 5.4. *Let V be a finite dimensional $U_q(\mathfrak{gl}(n))$-module. The $U_q(\mathfrak{gl}(n))$-intertwiners $\xi : (V \otimes V^*)^{op} \to V_\kappa$ are in bijective correspondence with $U_q(\mathfrak{gl}(n))$-intertwiners $\xi^* : V \to V \otimes V_\kappa$, where $V \otimes V_\kappa$ is now considered as $U_q(\mathfrak{gl}(n))$-module by*

$$X(v \otimes w) = \sum X_1 v \otimes X_2 w, \qquad X \in U_q(\mathfrak{gl}(n)),\, v \in V,\, w \in V_\kappa.$$

The bijective correspondence is explicitly given by the formula

$$(\phi \otimes Id)\xi^*(v) = \xi(v \otimes K^{2\delta_n}\phi), \qquad \phi \in V^*,\, v \in V$$

where (recall) $K^{2\delta_n} = K_1^{2(n-1)} K_2^{2(n-2)} \cdots K_{n-1}^2 \in U_q(\mathfrak{gl}(n))$.

Proof. The standard proof is left to the reader. $\square$

Let V be a finite dimensional $U_q(\mathfrak{gl}(n))$-module with basis $\{v_i\}$ and corresponding dual basis $\{v_i^*\}$. Let $\xi : (V \otimes V^*)^{op} \to V_\kappa$ be an $U_q(\mathfrak{gl}(n))$-intertwiner. Then

$$g_{V,\xi}(\cdot) = \sum_i \xi(\cdot\,(v_i \otimes v_i^*))$$

defines an element in $\widetilde{F}_\kappa$. Any element in $\widetilde{F}_\kappa$ is of this form for some V and ξ.

Lemma 5.5. *The radial part $f|_T \in \mathbb{C}[u^{\pm 1}]$ of a vector valued spherical function $f \in F_{\vec{\kappa}}^{\sigma,\tau}$ is divisible by*

$$I = u^{\delta(\kappa,0)} \prod_{1 \le i < j \le n} \left(q^2 u_i^{-1} u_j; q^2\right)_\kappa$$

in $\mathbb{C}[u^{\pm 1}]$. Moreover, the quotient $f|_T / I \in \mathbb{C}[u^{\pm 1}]$ is S_n-invariant.

Proof. Let $f \in F_{\vec{\kappa}}^{\sigma,\tau}$ be a vector valued spherical function and set $g = f \circ (\mathrm{Id} \otimes \psi) \in \widetilde{F}_\kappa$. Then $g = g_{V,\xi}$ for some finite dimensional $U_q(\mathfrak{gl}(n))$-module V and some $U_q(\mathfrak{gl}(n))$-intertwiner $\xi : (V \otimes V^*)^{op} \to V_\kappa$. Let $\lambda \in P_{2n}$, then

$$f|_T(q^\lambda) = f(K^{\lambda - \delta_{2n}}) = g(K^{\lambda^\ddagger - (1^n)} \otimes K^{2\delta_n})$$

with

$$\lambda^\ddagger = (\lambda_1 - \lambda_{2n}, \lambda_2 - \lambda_{2n-1}, \ldots, \lambda_n - \lambda_{n+1}) \in P_n. \tag{5.2}$$

By the previous lemma we thus obtain

$$\begin{aligned} f|_T(q^\lambda) &= \sum_i \xi\big(K^{\lambda^\ddagger-(1^n)}v_i \otimes K^{2\delta_n}v_i^*\big) \\ &= \sum_i (v_i^*\otimes \mathrm{Id})\xi^*(K^{\lambda^\ddagger-(1^n)}v_i) \\ &= \mathrm{Tr}|_V\big(\xi^*(K^{\lambda^\ddagger-(1^n)}(\,\cdot\,))\big). \end{aligned}$$

The results of Etingof and Kirillov [3] imply

$$\mathrm{Tr}|_V\big(\xi^*(K^\mu(\,\cdot\,))\big) = I(q^\mu)p(q^\mu), \qquad \forall \mu \in P_n$$

with $q^\mu = (q^{\mu_1}, q^{\mu_2}, \ldots, q^{\mu_n})$ and $p \in \mathbb{C}[u^{\pm 1}]^{S_n}$. Since $I \in \mathbb{C}[u^{\pm1}]$ is a homogeneous Laurent polynomial, we arrive at the desired result. □

We continue our analysis of the ground state for $\kappa_1 = \kappa_2 = 0$ following the method of Kirillov [7]. We express the elements B_j $(j \neq n)$ and B_n^σ of $\mathcal{A}_\sigma$ by

$$B_j = \widetilde{y}_j + \widetilde{x}_{2n-j}, \qquad B_n^\sigma = \widetilde{y}_n + \widetilde{x}_n + \vartheta_0(q^\sigma)K_n^{-2}$$

with

$$\widetilde{x}_j = K_{2n-j}^{-1}x_jK_j^{-1}, \qquad \widetilde{y}_j = y_jK_{j+1}^{-1}K_{2n-j}^{-1},$$

and with ϑ_l given by

$$\vartheta_l(s) = \frac{s^{-1}-sq^{-2l}}{q-q^{-1}}, \qquad l \in \mathbb{Z}, \tag{5.3}$$

cf. Lemma 4.3. We define elements $E_{n-1}, E_n, F_{n-1}, F_n \in U_q(\mathfrak{g})$ by

$$\begin{aligned} E_{n-1} &= \widetilde{x}_{n+1}, \qquad & E_n &= \widetilde{x}_n\widetilde{x}_{n+1} - \widetilde{x}_{n+1}\widetilde{x}_n, \\ F_{n-1} &= \widetilde{y}_{n-1}, \qquad & F_n &= \widetilde{y}_n\widetilde{y}_{n-1} - q^{-2}\widetilde{y}_{n-1}\widetilde{y}_n. \end{aligned}$$

Direct computations lead to the following lemma.

Lemma 5.6. **(i)** *For any σ,*

$$\begin{aligned} E_n &= B_n^\sigma E_{n-1} - E_{n-1}B_n^\sigma, \\ F_n &= B_n^\sigma F_{n-1} - q^{-2}F_{n-1}B_n^\sigma. \end{aligned}$$

(ii) *The following commutation relations hold in $U_q(\mathfrak{g})$,*

$$\begin{aligned} &E_{n-1}F_{n-1} = q^2F_{n-1}E_{n-1}, \qquad E_{n-1}F_n = q^2F_nE_{n-1}, \\ &E_nF_{n-1} = F_{n-1}E_n, \qquad E_nF_n = q^2F_nE_n + (q^{-2}-1)F_{n-1}E_{n-1}C_n^{-2}. \end{aligned}$$

For an ordered r-tuple $J = (j_1, j_2, \ldots, j_r)$ with $j_s \in \{n-1, n\}$ and $r \leq \kappa$ we write

$$F_J = F_{j_1}F_{j_2}\cdots F_{j_r}, \qquad E_J = E_{j_r}\cdots E_{j_2}E_{j_1}$$

and

$$r_{J,n-1} = \#\{s \mid j_s = n-1\}, \qquad r_{J,n} = r - r_{J,n-1}.$$

We furthermore define the Laurent polynomial $\Delta_J \in \mathbb{C}[u^{\pm 1}] \subset \mathbb{C}[z^{\pm 1}]$ by

$$\Delta_J = \prod_{s,t} \left(1 - q^{2s^2+2st+2t^2}(u_{n-1}^{-1}u_n)^s(u_{n-1}^{-1}u_n^{-1})^t\right),$$

where the product is taken over $(s,t) \in \mathbb{Z}_{\geq 0}^{\times 2}$ with $0 < s+t \leq r$ and $t \leq r_{J,n}$. For $J = \emptyset$, we write $F_\emptyset = E_\emptyset = 1$, $r_{\emptyset,n-1} = r_{\emptyset,n} = 0$ and $\Delta_\emptyset = 1$. Using the notations of Lemma 1.13, we can now formulate the following result.

Lemma 5.7. *Let $f \in F^{\sigma,\tau}_{(0,0,\kappa)}$ be a vector valued spherical function. Let $r \in \mathbb{Z}_{\geq 0}$ with $r \leq \kappa$ and fix an ordered r-tuple J with coefficients from $\{n-1, n\}$. Then there exists a Laurent polynomial $P_J \in \mathbb{C}[z^{\pm 1}]$ such that*

$$\Delta_J(q^\lambda) f(F_J K^{\lambda-\delta}) = P_J(q^\lambda) f|_T(q^\lambda) r_{(\kappa^{n-2},\kappa-r,\kappa+r)}$$

for all $\lambda \in P_{2n}$.

Proof. First note that the four elements $Y = E_{n-1}, E_n, F_{n-1}, F_n$ all have the same restricted weight,

$$C_j Y = q^{\delta_{j,n}-\delta_{j,n-1}} Y C_j, \qquad \forall\, j = 1, \ldots, n.$$

In particular, we conclude that

$$f(F_J K^{\lambda-\delta}) \in \operatorname{span}\{r_{(\kappa^{n-2},\kappa-r,\kappa+r)}\},$$

cf. Lemma 1.13, hence

$$f(F_J K^{\lambda-\delta}) = f_J(q^\lambda) r_{(\kappa^{n-2},\kappa-r,\kappa+r)}, \qquad \forall\, \lambda \in P_{2n}$$

for a unique Laurent polynomial $f_J \in \mathbb{C}[z^{\pm 1}]$. Furthermore, we may and will view all formulas below as identities in $\mathbb{C}[z^{\pm 1}]$ by identifying $\mathbb{C}r_{(\kappa^{n-2},\kappa-r,\kappa+r)} \simeq \mathbb{C}$.

We thus need to prove the existence of a Laurent polynomial $P_J \in \mathbb{C}[z^{\pm 1}]$ such that $\Delta_J f_J = P_J f|_T$. To avoid unnecessary technicalities we avoid the use of the underlying root data as much as possible. Let $\mathcal{F}_r$ be the finite set of r-tuples $J = (j_1, j_2, \ldots, j_r)$ with $j_s \in \{n-1, n\}$, and write $\mathcal{F}^\kappa = \cup_{0 \leq r \leq \kappa} \mathcal{F}_r$. For $I, J \in \mathcal{F}^\kappa$ we write $I < J$ if $\#I \lneq \#J$ and $r_{I,n} \leq r_{J,n}$, or if $\#I = \#J$ and $r_{I,n} \lneq r_{J,n}$. We prove the lemma by induction to $J \in \mathcal{F}^\kappa$ along $<$. For the smallest element $J = \emptyset \in \mathcal{F}_0$, we have $\Delta_\emptyset = 1$, $f_\emptyset = f|_T$, hence we can take $P_\emptyset = 1$. Suppose there exists a Laurent polynomial P_J such that $\Delta_J f_J = P_J f|_T$ for all $J \in \mathcal{F}^\kappa$ with $J < I = (i_1, i_2, \ldots, i_r) \in \mathcal{F}_r$. For simplicity we write $r_1 = r_{I,n-1}$ and $r_2 = r_{I,n}$.

We write $g \equiv g'$ for $g, g' \in \mathbb{C}[z^{\pm 1}]$ if $g - g'$ lies in the ideal of $\mathbb{C}[z^{\pm 1}]$ generated by the f_J's for $J \in \mathcal{F}^\kappa$ with $J < I$. Sometimes it is convenient to write $g(q^\lambda) \equiv g'(q^\lambda)$, where we have formally evaluated the equivalence $g \equiv g'$ of the Laurent polynomials g and g' in $z = q^\lambda$ for arbitrary $\lambda \in P_{2n}$.

To prove the existence of the Laurent polynomial P_I satisfying $\Delta_I f_I = P_I f|_T$, it suffices to show that

$$\left(1 - q^{2r_1^2+2r_1r_2+2r_2^2}(u_{n-1}^{-1}u_n)^{r_1}(u_{n-1}^{-1}u_n^{-1})^{r_2}\right) f_I \equiv 0 \tag{5.4}$$

due to the induction hypothesis and the fact that Δ_J divides Δ_I in $\mathbb{C}[u^{\pm 1}] \subset \mathbb{C}[z^{\pm 1}]$ when $J < I$. We start by noting that

$$\begin{aligned} &f(F_{n-1}X) = -f(E_{n-1}X) + \rho(\kappa,0)_\tau(B_{n-1})\big(f(X)\big),\\ &f(F_nX) = -q^{-2}f(E_nX) + (1-q^{-2})\rho(\kappa,0)_\tau(B_n^\tau)\big(f(F_{n-1}X)\big) + \rho(\kappa,0)_\tau([B_n^\tau, B_{n-1}])\big(f(X)\big) \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} f(XE_{n-1}) &= -f(XF_{n-1}),\\ f(XE_n) &= -f(XF_n) + (1-q^{-2})\vartheta_0(q^\sigma)f(XF_{n-1}) \end{aligned} \tag{5.6}$$

for any $X \in U_q(\mathfrak{g})$, which follows easily from Lemma 5.6**(i)** and the fact that f is a vector valued spherical function. Denote $I' = (i_2, i_3, \ldots, i_r)$, which is $< I$. Then it follows from (5.5) that

$$f_I(q^\lambda) \equiv -q^{-2\delta_{i_1,n}} f(E_{i_1}F_{I'}K^{\lambda-\delta}).$$

By Lemma 5.6**(ii)** we now conclude that

$$f_I(q^\lambda) \equiv \begin{cases} -q^{2(r-1)}f(F_{I'}E_{i_1}K^{\lambda-\delta}), & \text{if } \ i_1 = n-1,\\ -q^{-2}q^{2(r_2-1)}f(F_{I'}E_{i_1}K^{\lambda-\delta}), & \text{if } \ i_1 = n. \end{cases}$$

Continuing inductively while taking (5.6) into account, we obtain

$$\begin{aligned} f_I(q^\lambda) &\equiv (-1)^r q^{-2r_2 + r_2(r_2-1) + 2\sum_{s:i_s=n-1}(r-s)} f(E_I K^{\lambda-\delta})\\ &\equiv (-1)^r q^{r_1 + r_2(r_2-1) + 2\sum_{s:i_s=n-1}(r-s)} q^{-r_2\lambda_n - r_1\lambda_{n+1} + r\lambda_{n+2}} f(K^{\lambda-\delta}E_I). \end{aligned} \tag{5.7}$$

We follow a similar procedure to return from $f(K^{\lambda-\delta}E_I)$ to $f_I(q^\lambda)$. The starting point is

$$f(K^{\lambda-\delta}E_I) \equiv -f(K^{\lambda-\delta}E_{I'}F_{i_1}),$$

which follows from (5.6). Moving F_{i_1} to the left using Lemma 5.6**(ii)** gives

$$f(K^{\lambda-\delta}E_I) \equiv \begin{cases} -q^{2(r_1-1)}f(K^{\lambda-\delta}F_{i_1}E_{I'}), & \text{if } \ i_1 = n-1,\\ -q^{2(r-1)}f(K^{\lambda-\delta}F_{i_1}E_{I'}), & \text{if } \ i_1 = n. \end{cases}$$

Repeating this procedure inductively gives

$$\begin{aligned} f(K^{\lambda-\delta}E_I) &\equiv (-1)^r q^{r_1(r_1-1) + 2\sum_{s:i_s=n}(r-s)} f(K^{\lambda-\delta}F_I)\\ &\equiv (-1)^r q^{r_1^2 + 2r_2 + 2\sum_{s:i_s=n}(r-s)} q^{-r\lambda_{n-1} + r_1\lambda_n + r_2\lambda_{n+1}} f_I(q^\lambda). \end{aligned} \tag{5.8}$$

Combining (5.7) and (5.8) yields the induction step (5.4). □

Lemma 5.8. *The radial part $f|_T \in \mathbb{C}[u^{\pm 1}]$ of a vector valued spherical function $f \in F^{\sigma,\tau}_{(0,0,\kappa)}$ is divisible by $(1 - q^{2r}u_{n-1}^{-1}u_n^{-1})$ for $r = 1, \ldots, \kappa$.*

Proof. It is easy to check that $\Delta_{(n^\kappa)}$ is divisible by $1 - q^{2r}u_{n-1}^{-1}u_n^{-1}$ in $\mathbb{C}[u^{\pm 1}]$ for $r = 1, \ldots, \kappa$. By the previous lemma it thus suffices to show that $P_{(n^\kappa)}$ is nonzero and relative prime to $1 - q^{2r}u_{n-1}^{-1}u_n^{-1}$ for $r = 1, \ldots, \kappa$. Clearly it is enough to prove this in the classical limit $q = 1$.

We thus consider $q = e^h$ as a formal parameter and we repeat part of the computations of the proof of Lemma 5.7 for $J = (n^r)$, now modding out to $h\mathbb{C}[[h]][z^{\pm 1}]$.

We simplify notations by writing

$$f(F_n^s K^{\lambda-\delta}) = f_s(q^\lambda) r_{(\kappa^{n-2},\kappa-s,\kappa+s)},$$
$$f(E_n F_n^{s-1} K^{\lambda-\delta}) = g_s(q^\lambda) r_{(\kappa^{n-2},\kappa-s,\kappa+s)}$$

for all $\lambda \in P_{2n}$ and $s \in \{1, \ldots, \kappa\}$, with $f_s, g_s \in \mathbb{C}[[h]][z^{\pm 1}]$. Lemma 1.13 gives

$$\rho(\kappa,0)_\tau([B_n^\tau, B_{n-1}]) f(F_n^{r-1} K^{\lambda-\delta}) = q^{2-\tau-3r}[r-1-\kappa]_q f_{r-1}(q^\lambda) r_{(\kappa^{n-2},\kappa-r,\kappa+r)},$$

with

$$[\alpha]_q = \frac{q^\alpha - q^{-\alpha}}{q - q^{-1}}$$

the (symmetric) q-number. Identifying $\mathbb{C} r_{(\kappa^{n-2},\kappa-r,\kappa+r)} \simeq \mathbb{C}$ as in the proof of Lemma 5.7, we obtain from the second formula of (5.5),

$$f_r = -q^{-2} g_{r-1} + q^{2-\tau-3r}[r-1-\kappa]_q f_{r-1} \qquad \mathrm{mod}\, h.$$

By Lemma 5.6**(ii)**, we can move E_n to the other side in the expression

$$g_{r-1}(q^\lambda) = f(E_n F_n^{r-1} K^{\lambda-\delta}),$$

after which we can use the second formula in (5.6) to replace E_n by $-F_n$ modulo h. This yields

$$g_{r-1} = -q^{2(r+1)} u_{n-1}^{-1} u_n^{-1} f_r \qquad \mathrm{mod}\, h.$$

Consequently,

$$(1 - q^{2r} u_{n-1}^{-1} u_n^{-1}) f_r = q^{2-\tau-3r}[r-1-\kappa]_q f_{r-1} \qquad \mathrm{mod}\, h,$$

or, more precisely,

$$(1 - u_{n-1}^{-1} u_n^{-1}) f_r = (r-1-\kappa) f_{r-1} \qquad \mathrm{mod}\, h$$

for $r = 1, \ldots, \kappa$. Thus $P_{(n^\kappa)}$ is nonzero, and

$$P_{(n^\kappa)} = C(1 - u_{n-1}^{-1} u_n^{-1})^{-\kappa} \Delta_{(n^\kappa)} \qquad \mathrm{mod}\, h, \tag{5.9}$$

for some nonzero constant $C \in \mathbb{C}$. Using the explicit expression of $\Delta_{(n^\kappa)}$, it is easy to check that the right hand side of (5.9) (mod h) is relative prime to $1 - u_{n-1}^{-1} u_n^{-1}$ in $\mathbb{C}[z^{\pm 1}]$, which completes the proof of the lemma. □

We are now in a position to determine the explicit form of the radial part of the ground state when $\kappa_1 = \kappa_2 = 0$.

Corollary 5.9. *Let $f \in F^{\sigma,\tau}_{(0,0,\kappa)}(0)$ be a ground state. With a suitable normalization of f, we have*

$$f|_T = u^{\delta(\kappa,0)} \prod_{1 \le i < j \le n} \left(q^2 u_i^{-1} u_j, q^2 u_i^{-1} u_j^{-1}; q^2\right)_\kappa \tag{5.10}$$

in $\mathbb{C}[u^{\pm 1}]$.

Proof. We write $\mathcal{I} \in \mathbb{C}[u^{\pm 1}]$ for the right hand side of (5.10). Lemma 5.5 and Lemma 5.8 show that $f|_T$ is divisible by $\mathcal{I}$ in $\mathbb{C}[u^{\pm 1}]$, so $f|_T = \mathcal{I}p$ for some Laurent polynomial $p \in \mathbb{C}[u^{\pm 1}]$. It remains to show that p is a nonzero constant. For this we expand both sides of the equality $f|_T = \mathcal{I}p$ in monomials u^μ ($\mu \in \Lambda_n$).

We write

$$p = \sum_\nu e_\mu u^\nu,$$

and we denote $\nu_+ \in \Lambda_n$ (respectively $\nu_- \in \Lambda_n$) for a maximal (respectively minimal) element with respect to the dominance order $\leq$ such that $e_{\nu_+} \neq 0 \neq e_{\nu_-}$. Observe that

$$\mathcal{I} = \sum_{-\delta(\kappa,0) \leq \nu \leq \delta(\kappa,0)} c_\nu u^\nu$$

with $c_\nu \in \mathbb{C}$ and $c_{-\delta(\kappa,0)} \neq 0 \neq c_{\delta(\kappa,0)}$. Consequently, in the expansion of $\mathcal{I}p$ in monomials u^μ ($\mu \in \Lambda_n$), the coefficient of $u^{\delta(\kappa,0)+\nu_+}$ and of $u^{-\delta(\kappa,0)-\nu_-}$ is nonzero.

On the other hand, Lemma 4.1 implies that

$$f|_T = \sum_{-\delta(\kappa,0) \leq \nu \leq \delta(\kappa,0)} d_\nu u^\nu$$

with $d_\nu \in \mathbb{C}$ and $d_{-\delta(\kappa,0)} \neq 0 \neq d_{\delta(\kappa,0)}$. Hence the equality $f|_T = \mathcal{I}p$ forces $\nu_+ \leq 0$ and $\nu_- \geq 0$. This in turn implies $\nu_+ = \nu_- = 0$, hence p is a nonzero constant. $\square$

5.3. **The ground state for $\kappa = 0$.** We start with the following preliminary lemma, which will be convenient for several computations in this subsection. Recall the definition of $\vartheta_k(s)$, see (5.3).

Lemma 5.10. *Let W_1 and W_2 be left $U_q(\mathfrak{g})$-modules and let $k \in \mathbb{Z}$.*

(i) *If $w_0 \in W_1$ is a vector satisfying $B_n^\sigma w_0 = \vartheta_0(q^{\sigma+k})w_0$ and $C_n w_0 = w_0$, then*

$$B_n^\sigma(w_0 \otimes w') = w_0 \otimes B_n^{\sigma+k}w', \qquad \forall\, w' \in W_2$$

in the $U_q(\mathfrak{g})$-module $W_1 \otimes W_2$.

(ii) *If $w_\pm \in W_1$ is a vector satisfying $B_n^\sigma w_\pm = \vartheta_k(\pm q^{\pm\sigma})w_\pm$ and $C_n w_\pm = q^k w_\pm$, then*

$$B_n^\sigma(w_\pm \otimes w') = w_\pm \otimes q^{-k}B_n^{\sigma\mp k}w', \qquad \forall\, w' \in W_2$$

in the $U_q(\mathfrak{g})$-module $W_1 \otimes W_2$.

Proof. Recall that the generator $B_n^\sigma \in \mathcal{A}_\sigma$ is explicitly given by

$$B_n^\sigma = y_n K_{n+1}^{-1}K_n^{-1} + K_n^{-1}x_n K_n^{-1} + \vartheta_0(q^\sigma)K_n^{-2}.$$

The proof of **(i)** follows now immediately from the explicit formula for $\Delta(B_n^\sigma)$ (see Proposition 4.2) and from the fact that $C_n = C_{n+1}$. For the proof of **(ii)**, let $w_\pm \in W_1$ be as indicated in the lemma, and let $w' \in W_2$. Then Proposition 4.2 implies

$$\begin{aligned} B_n^\sigma\big(w_\pm \otimes w'\big) &= w_\pm \otimes q^{-k}\big(y_n K_{n+1}^{-1}K_n^{-1} + K_n^{-1}x_n K_n^{-1} + q^k\vartheta_k(\pm q^{\pm\sigma})K_n^{-2}\big)w' \\ &= w_\pm \otimes q^{-k}B_n^{\sigma\mp k}w' \end{aligned}$$

since $q^k\vartheta_k(\pm q^{\pm\sigma}) = \vartheta_0(q^{\sigma\mp k})$. $\square$

In this subsection the realization of the fundamental, simple $U_q(\mathfrak{g})$-modules within the q-exterior algebra of the vector representation of $U_q(\mathfrak{g})$ is used to explicitly construct the one-dimensional $\mathcal{A}_\sigma$-modules $V(\kappa_1)_\sigma$ and $V(\kappa_2)_\sigma$ within simple $U_q(\mathfrak{g})$-modules. As a consequence we obtain an explicit expression for the radial part of the ground state for $\kappa = 0$.

The vector representation of $U_q(\mathfrak{g})$ is the $2n$-dimensional vector space V with linear basis v_i $(i = 1, \ldots, 2n)$ and with $U_q(\mathfrak{g})$-action defined by

$$K_r^{\pm 1} v_j = q^{\pm \delta_{r,j}} v_j, \qquad x_i v_j = \delta_{i+1,j}\, v_{j-1}, \qquad y_i v_j = \delta_{i,j}\, v_{j+1} \tag{5.11}$$

for $r, j = 1, \ldots, 2n$ and $i = 1, \ldots, 2n-1$, with the convention that $v_0 = v_{2n+1} = 0$. This is a realization of the finite dimensional irreducible highest weight module L_{ϵ_1} of $U_q(\mathfrak{g})$, with highest weight vector v_1. Similarly, the dual vector representation is the $2n$-dimensional vector space V^* with basis v_i^* $(i = 1, \ldots, 2n)$ and with $U_q(\mathfrak{g})$-action given explicitly by

$$K_r^{\pm 1} v_j^* = q^{\mp \delta_{r,j}} v_j^*, \qquad x_i v_j^* = -q^{-1} \delta_{i,j}\, v_{j+1}^*, \qquad y_i v_j^* = -q \delta_{i+1,j}\, v_{j-1}^*$$

for $r, j = 1, \ldots, 2n$ and $i = 1, \ldots, 2n-1$, with the convention that $v_0^* = v_{2n+1}^* = 0$ (it corresponds to the linear dual V^* of V viewed as $U_q(\mathfrak{g})$-module by $(X\phi)(v) = \phi(S(X)v)$ for $\phi \in V^*$, $v \in V$ and $X \in U_q(\mathfrak{g})$). This is a realization of the finite dimensional irreducible highest weight module $L_{-\epsilon_{2n}}$ of $U_q(\mathfrak{g})$, with highest weight vector v_{2n}^*. For $s \in \mathbb{R} \setminus \{0\}$ we define n-dimensional subspaces $W(s) = \operatorname{span}\{w_i(s)\}_i \subseteq V$ and $W^*(s) = \operatorname{span}\{w_i^*(s)\}_i \subseteq V^*$ by

$$w_i(s) = v_i + s^{-1} v_{2n+1-i}, \qquad\qquad w_i^*(s) = v_i^* + s^{-1} q^{2i-2n-1}\, v_{2n+1-i}^*$$

for $i = 1, \ldots, n$.

Proposition 5.11. *The vector spaces $W(\pm q^{\pm\sigma})$ (respectively $W^*(\pm q^{\pm\sigma})$) are inequivalent simple $\mathcal{A}_\sigma$-submodules of V^σ (respectively $V^{*\sigma}$). The irreducible decompositions of the semisimple $\mathcal{A}_\sigma$-modules V^σ and $V^{*\sigma}$ are*

$$V^\sigma = W(q^\sigma) \oplus W(-q^{-\sigma}), \qquad\qquad V^{*\sigma} = W^*(q^\sigma) \oplus W^*(-q^{-\sigma}).$$

Proof. The action of the algebraic generators of $\mathcal{E} \subset \mathcal{A}_\sigma$ (see Lemma 1.8) on $w_i(s)$ and $w_i^*(s)$ is given by the formulas

$$\begin{aligned}
C_j w_i(s) &= q^{\delta_{i,j}} w_i(s), & C_j w_i^*(s) &= q^{-\delta_{i,j}} w_i^*(s) & & (i, j = 1, \ldots, n),\\
B_i w_i(s) &= w_{i+1}(s), & B_i w_{i+1}^*(s) &= -q^2 w_i^*(s) & & (i = 1, \ldots, n-1),\\
B_{2n+1-i}\, w_i(s) &= w_{i-1}(s), & B_{2n+1-i}\, w_{i-1}^*(s) &= -w_i^*(s) & & (i = 2, \ldots, n),\\
B_j w_i(s) &= 0, & B_j w_i^*(s) &= 0 & & \text{otherwise.}
\end{aligned} \tag{5.12}$$

Hence $W(s)$ (respectively $W^*(s)$) is a $\mathcal{E}$-submodule of V (respectively V^*) which is isomorphic to the vector representation (respectively dual vector representation) of $U_q(\mathfrak{gl}(n)) \simeq \mathcal{E}$, cf. Lemma 1.8. In particular, $W(s)$ and $W^*(s)$ are simple $\mathcal{E}$-modules.

Recall the notation (5.3). By a direct computation one verifies that

$$B_n^\sigma w_i(s) = \vartheta_0(q^\sigma)\, w_i(s), \qquad\qquad B_n^\sigma w_i^*(s) = \vartheta_0(q^\sigma)\, w_i^*(s) \tag{5.13}$$

for all $i = 1, \ldots, n-1$, and

$$B_n^\sigma w_n(\pm q^{\pm\sigma}) = \vartheta_1(\pm q^{\pm\sigma})\, w_n(\pm q^{\pm\sigma}), \qquad\qquad B_n^\sigma w_n^*(\pm q^{\pm\sigma}) = \vartheta_{-1}(\pm q^{\pm\sigma})\, w_n^*(\pm q^{\pm\sigma}). \tag{5.14}$$

It is now clear that $W(\pm q^{\pm\sigma})\subset V^\sigma$ are simple $\mathcal{A}_\sigma$-submodules, and that $V^\sigma=W(q^\sigma)\oplus W(-q^{-\sigma})$ (and similarly for $V^{*\sigma}$). To prove that $W(q^\sigma)\not\simeq W(-q^{-\sigma})$ (resp. $W^*(q^\sigma)\not\simeq W^*(-q^{-\sigma})$) as $\mathcal{A}_\sigma$-modules, it suffices to observe that the spectrum of B_n^σ is different. This follows from the fact that $\vartheta_k(q^\sigma)=\vartheta_k(-q^{-\sigma})$ for $\sigma\in\mathbb{R}$ implies $k=0$. □

We now consider the q-exterior algebras of V and V^*, cf. [21]. Let $T(V)$ and $T(V^*)$ be the tensor algebras of V and V^*, respectively. Let $J\subset T(V)$ be the graded, two-sided ideal generated by the tensors $v_i\otimes v_i$ and $v_r\otimes v_s+q^{-1}v_s\otimes v_r$ for $i=1,\ldots,2n$ and $1\leq r<s\leq 2n$. Similarly, we write $J_*\subset T(V^*)$ for the graded, two-sided ideal generated by the tensors $v_i^*\otimes v_i^*$ and $v_s^*\otimes v_r^*+q^{-1}v_r^*\otimes v_s^*$ for $i=1,\ldots,2n$ and $1\leq r<s\leq 2n$. The exterior algebras $\Lambda(V)$ and $\Lambda(V^*)$ are defined to be the graded algebras $T(V)/J$ and $T(V^*)/J_*$, respectively. The action of $U_q(\mathfrak{g})$ on V and V^* naturally extends to a grading preserving, left $U_q(\mathfrak{g})$-module algebra structure on the exterior algebras $\Lambda(V)$ and $\Lambda(V^*)$. We denote $\wedge$ for the products in $\Lambda(V)$ and $\Lambda(V^*)$. Let $\Lambda^m(V)$ and $\Lambda^m(V^*)$ be the mth graded pieces of $\Lambda(V)$ and $\Lambda(V^*)$ respectively, then

$$\Lambda(V)=\bigoplus_{m=0}^{2n}\Lambda^m(V),\qquad \Lambda(V^*)=\bigoplus_{m=0}^{2n}\Lambda^m(V^*)$$

and

$$\Lambda^m(V)\simeq L_{(1^m,0^{2n-m})},\qquad \Lambda^m(V^*)\simeq L_{(0^{2n-m},-1^m)}$$

as $U_q(\mathfrak{g})$-modules. For $I=\{i_1,\ldots,i_m\}\subseteq\{1,\ldots,2n\}$ with $i_1<i_2<\cdots<i_m$ we define

$$v_I=v_{i_1}\wedge v_{i_2}\wedge\cdots\wedge v_{i_m},\qquad v_I^*=v_{i_1}^*\wedge v_{i_2}^*\wedge\cdots\wedge v_{i_m}^*.$$

Then $\{v_I\,|\,\#I=m\}$ and $\{v_I^*\,|\,\#I=m\}$ are linear bases of $\Lambda^m(V)$ and $\Lambda^m(V^*)$, respectively. The action of $U_q(\mathfrak{g})$ on these basis elements is explicitly given by

$$\begin{aligned}
&K_i^{\pm1}v_I=q^{\pm\delta(i,I)}v_I, && x_iv_I=v_{(I\setminus\{i+1\})\cup\{i\}}, && y_iv_I=v_{(I\setminus\{i\})\cup\{i+1\}},\\
&K_i^{\pm1}v_I^*=q^{\mp\delta(i,I)}v_I^*, && x_iv_I^*=-q^{-1}v^*_{(I\setminus\{i\})\cup\{i+1\}}, && y_iv_I^*=-qv^*_{(I\setminus\{i+1\})\cup\{i\}},
\end{aligned}$$

where $\delta(i,I)=1$ if $i\in I$ and $=0$ otherwise, and where $v_{(I\setminus\{k\})\cup\{l\}}=0$ and $v^*_{(I\setminus\{k\})\cup\{l\}}=0$ if $k\notin I$ or if $l\in I$. Note in particular that $v_{\{1,\ldots,m\}}\in\Lambda^m(V)$ and $v^*_{\{2n-m+1,\ldots,2n\}}\in\Lambda^m(V^*)$ are highest weight vectors. The formulas

$$\langle v_I,v_J\rangle=\delta_{I,J},\qquad \langle v_I^*,v_J^*\rangle_*=q^{2\sum_{i\in I}i}\,\delta_{I,J}$$

define $*$-unitary scalar products on $\Lambda(V)$ and $\Lambda(V^*)$, respectively.

Using the notation (2.1), we observe that any one-dimensional $\mathcal{A}_\sigma$-module in $\Lambda^n(V)^\sigma$ (resp. $\Lambda^n(V^*)^\sigma$) necessarily lies in the subspace $\Lambda^n(V)_{(1^n)}$ (resp. $\Lambda^n(V^*)_{(-1^n)}$). This follows easily using Lemma 1.8, since an one-dimensional $U_q(\mathfrak{gl}(n))$-module is of highest weight $(m^n)\in P_n^+$ for some integer m. For $\underline{s}=(s_1,\ldots,s_n)$ an n-tuple of nonzero reals, we define $\xi(\underline{s})\in\Lambda^n(V)_{(1^n)}$ and $\xi^*(\underline{s})\in\Lambda^n(V^*)_{(-1^n)}$ by

$$\begin{aligned}
\xi(\underline{s})&=w_1(s_1)\wedge w_2(s_2)\wedge\cdots\wedge w_n(s_n),\\
\xi^*(\underline{s})&=w_1^*(s_1)\wedge w_2^*(s_2)\wedge\cdots\wedge w_n^*(s_n).
\end{aligned}$$

It is also convenient to use the opposite elements $\xi_{op}(\underline{s}) \in \Lambda^n(V)_{(1^n)}$ and $\xi^*_{op}(\underline{s}) \in \Lambda^n(V^*)_{(-1^n)}$, defined by

$$\begin{aligned}\xi_{op}(\underline{s}) &= w_n(s_n) \wedge \cdots \wedge w_2(s_2) \wedge w_1(s_1),\\ \xi^*_{op}(\underline{s}) &= w^*_n(s_n) \wedge \cdots \wedge w^*_2(s_2) \wedge w^*_1(s_1).\end{aligned}$$

For $1 \le i < j \le n$ we have

$$\begin{aligned}w_i(s_i) \wedge w_j(s_j) &= -q^{-1} w_j(s_j) \wedge w_i(q^{-2}s_i),\\ w^*_i(s_i) \wedge w^*_j(s_j) &= -q w^*_j(s_j) \wedge w^*_i(q^2 s_i)\end{aligned}$$

in $\Lambda(V)$ (resp. $\Lambda(V^*)$), hence ξ and ξ_{op} (resp. ξ^* and ξ^*_{op}) are related by the formulas

$$\begin{aligned}\xi(q^{2(n-1)}s_1, \ldots, q^2 s_{n-1}, s_n) &= \big(-q\big)^{n(1-n)/2} \xi_{op}(\underline{s}),\\ \xi^*(q^{-2(n-1)}s_1, \ldots, q^{-2} s_{n-1}, s_n) &= \big(-q\big)^{n(n-1)/2} \xi^*_{op}(\underline{s}).\end{aligned} \tag{5.15}$$

The norms of such vectors can be evaluated explicitly by an easy induction argument. The result is

$$\begin{aligned}\langle \xi(\underline{s}), \xi(\underline{t})\rangle &= \prod_{i=1}^n \Big(1 + s_i^{-1} t_i^{-1} q^{2(n-i)}\Big),\\ \langle \xi^*(\underline{s}), \xi^*(\underline{t})\rangle_* &= q^{n(n+1)} \prod_{i=1}^n \Big(1 + s_i^{-1} t_i^{-1} q^{2(i-n)}\Big).\end{aligned} \tag{5.16}$$

We use the abbreviation $\xi(t) = \xi((t^n))$ for $t \in \mathbb{R} \setminus \{0\}$, where (t^n) is the n-tuple with t in each entry. Similarly we write $\xi_{op}(t)$, $\xi^*(t)$ and $\xi^*_{op}(t)$.

Lemma 5.12. **(i)** *Let $\underline{s} \in \big(\mathbb{R} \setminus \{0\}\big)^n$. Then $span_{\mathbb{C}}\{\xi_{op}(\underline{s})\} \subset \Lambda^n(V)$ and $span_{\mathbb{C}}\{\xi^*_{op}(\underline{s})\} \subset \Lambda^n(V^*)$ are one-dimensional $\mathcal{E}$-submodules if and only if $\underline{s} = (t^n)$ for some $t \in \mathbb{R} \setminus \{0\}$.*

(ii) *The two subspaces $span\{\xi_{op}(\pm q^{\pm\sigma})\} \subset \Lambda^n(V)^\sigma$ are one-dimensional $\mathcal{A}_\sigma$-submodules of $\Lambda^n(V)^\sigma$. The generator $B^\sigma_n \in \mathcal{A}_\sigma$ acts by*

$$B^\sigma_n\, \xi_{op}(\pm q^{\pm\sigma}) = \vartheta_1(\pm q^{\pm\sigma}) \xi_{op}(\pm q^{\pm\sigma}).$$

(iii) *The two subspaces $span\{\xi^*_{op}(\pm q^{\pm\sigma})\} \subset \Lambda^n(V^*)^\sigma$ are one-dimensional $\mathcal{A}_\sigma$-submodules of $\Lambda^n(V^*)^\sigma$. The generator $B^\sigma_n \in \mathcal{A}_\sigma$ acts by*

$$B^\sigma_n\, \xi^*_{op}(\pm q^{\pm\sigma}) = \vartheta_{-1}(\pm q^{\pm\sigma}) \xi^*_{op}(\pm q^{\pm\sigma}).$$

Proof. **(i)** Fix $i \in \{1, \ldots, n-1\}$. By (5.11) we have

$$x_{2n-i}\, w_j(s) = y_i\, w_j(s) = 0 \qquad \forall j \in \{1, \ldots, n\} \setminus \{i\}.$$

Combined with (5.12) and Proposition 4.2 we obtain

$$\begin{aligned}B_i\, \xi(\underline{s}) &= w_1(s_1) \wedge \cdots \wedge w_{i-1}(s_{i-1}) \wedge B_i\, w_i(s_i) \wedge K_i^{-1} K_{2n-i}^{-1}\big(w_{i+1}(s_{i+1}) \wedge \cdots \wedge w_n(s_n)\big)\\ &= w_1(s_1) \wedge \cdots \wedge w_{i-1}(s_{i-1}) \wedge w_{i+1}(s_i) \wedge w_{i+1}(q s_{i+1}) \wedge w_{i+2}(s_{i+2}) \wedge \cdots \wedge w_n(s_n).\end{aligned}$$

Since

$$w_{i+1}(s_i) \wedge w_{i+1}(q s_{i+1}) = (s_i^{-1} - q^{-2} s_{i+1}^{-1})\, v_{2n-i} \wedge v_{i+1}$$

in $\Lambda(V)$, we see that $B_i\,\xi(\underline{s})=0$ for all $i=1,\ldots,n-1$ if and only if $\underline{s}=(tq^{2(n-1)},\ldots,tq^2,t)$ for some $t\in\mathbb{R}\setminus\{0\}$. Furthermore, $C_j\xi(\underline{s})=q\xi(\underline{s})$ for $j=1,\ldots,n$ by (5.12). Then Lemma 1.8 and (5.15) imply that $\xi_{op}(\underline{s})$ is a lowest $\mathcal{E}\simeq U_q(\mathfrak{gl}(n))$-weight vector of weight (1^n) in $\Lambda^n(V)$ if and only if $\underline{s}=(t^n)$ for some $t\in\mathbb{R}\setminus\{0\}$, which implies **(i)** for ξ_{op}. The proof for ξ_{op}^* is similar.

For the proof of **(ii)** and **(iii)**, we note that Lemma 5.10**(ii)**, (5.12) and (5.13) imply

$$\begin{aligned} B_n^\sigma\,\xi(\underline{s})&=w_1(s_1)\wedge w_2(s_2)\wedge\cdots w_{n-1}(s_{n-1})\wedge B_n^\sigma\,w_n(s_n),\\ B_n^\sigma\,\xi^*(\underline{s})&=w_1^*(s_1)\wedge w_2^*(s_2)\wedge\cdots w_{n-1}^*(s_{n-1})\wedge B_n^\sigma\,w_n^*(s_n).\end{aligned}$$

Formula (5.14) now completes the proof. □

In the following proposition we consider the $U_q(\mathfrak{g})$-module $\Lambda^n(V)\otimes\Lambda^n(V^*)$. By the Pieri formula (cf. [2]) its decomposition in irreducibles is given by

$$\Lambda^n(V)\otimes\Lambda^n(V^*)\simeq\bigoplus_{m=0}^{n}L_{\omega_m^\natural},\tag{5.17}$$

where $\omega_m=(1^m,0^{n-m})\in\Lambda_n^+$. We denote $\pi_n:\Lambda^n(V)\otimes\Lambda^n(V^*)\to L_{\omega_n^\natural}$ for the projection onto $L_{\omega_n^\natural}$ along the decomposition (5.17).

Since $\delta(0,\kappa_1)=\kappa_1\omega_n$, (5.17) and our earlier analysis of branching rules easily imply that the $V(\pm1)_\sigma$-isotypical component of $(\Lambda^n(V)\otimes\Lambda^n(V^*))^\sigma$ is one-dimensional (and necessarily lies in the component $L_{\omega_n^\natural}$), while the $V(0)_\sigma$-isotypical component of $(\Lambda^n(V)\otimes\Lambda^n(V^*))^\sigma$ is $(n+1)$-dimensional. In [2], the $V(0)_\sigma$-isotypical component was constructed using the explicit solution J^σ of the reflection equation. In the following proposition we give an explicit construction of one particular vector in the $V(0)_\sigma$-isotypical component and we explicitly construct the $V(\pm1)_\sigma$-isotypical components of $(\Lambda^n(V)\otimes\Lambda^n(V^*))^\sigma$.

Proposition 5.13. *Define $\eta_k(s)\in\Lambda^n(V)\otimes\Lambda^n(V^*)$ ($s\in\mathbb{R}\setminus\{0\}$, $k\in\{-1,0,1\}$) by the formulas*

$$\begin{aligned}\eta_{-1}(s)&=\xi_{op}(s)\otimes\xi_{op}^*(-qs^{-1}),\\ \eta_0(s)&=\xi_{op}(s)\otimes\xi_{op}^*(q^{-1}s),\\ \eta_1(s)&=\xi_{op}(-s^{-1})\otimes\xi_{op}^*(qs).\end{aligned}$$

Then $\mathbb{C}\{\eta_k(q^\sigma)\}\subset\big(\Lambda^n(V)\otimes\Lambda^n(V^*)\big)^\sigma$ is a one-dimensional $\mathcal{A}_\sigma$-submodule, isomorphic to $V(k)_\sigma$. Furthermore, $\mathbb{C}\{\eta_k(q^\sigma)\}$ ($k=\pm1$) and $\pi_n(\mathbb{C}\{\eta_0(q^\sigma)\})$ are realizations of the one-dimensional $\mathcal{A}_\sigma$-modules $V(k)_\sigma$ ($k=\pm1$) and $V(0)_\sigma$ in $L^\sigma_{\omega_n^\natural}$, respectively.

Proof. By Proposition 4.2 and the previous lemma, $\mathbb{C}\{\xi_{op}(s)\otimes\xi_{op}^*(t)\}\subset\Lambda^n(V)\otimes\Lambda^n(V^*)$ is a copy of the trivial $\mathcal{E}\simeq U_q(\mathfrak{gl}(n))$ module for all $s,t\in\mathbb{R}\setminus\{0\}$. Furthermore, by Lemma 5.10 and the previous lemma,

$$B_n^\sigma(\xi_{op}(\pm q^{\pm\sigma})\otimes\xi_{op}^*(t))=\xi_{op}(\pm q^{\pm\sigma})\otimes q^{-1}B_n^{\sigma\mp1}(\xi_{op}^*(t))$$

in $\Lambda^n(V)\otimes\Lambda^n(V^*)$ for all $t\in\mathbb{R}\setminus\{0\}$. The first statement now follows from Lemma 5.12 and the equalities

$$q^{-1}\vartheta_{-1}(-q^{-\sigma+1})=\vartheta_0(q^{\sigma-2}),\qquad q^{-1}\vartheta_{-1}(q^{\sigma-1})=\vartheta_0(q^{\sigma}),\qquad q^{-1}\vartheta_{-1}(q^{\sigma+1})=\vartheta_0(q^{\sigma+2}),$$

which are the scalars by which B_n^σ acts on $V(-1)_\sigma$, $V(0)_\sigma$ and $V(1)_\sigma$, respectively.

We have already observed that $\mathbb{C}\{\eta_{\pm1}(q^\sigma)\}$ automatically lies in the component $L^\sigma_{\omega_n^\natural}$ of the decomposition (5.17). In the decomposition of $\eta_0(q^\sigma)$ as sum of basis elements $v_I\otimes v_J^*$ ($\#I=\#J=n$), the coefficient of $v_{\{1,\ldots,n\}}\otimes v^*_{\{n+1,\ldots,2n\}}$ is nonvanishing, hence $\pi_n(\eta_0(q^\sigma))\neq0$. This completes the proof of the second statement of the proposition. □

We define a $*$-unitary scalar product on $\Lambda^n(V)\otimes\Lambda^n(V^*)$ by

$$\langle v\otimes v^*,w\otimes w^*\rangle_1=\langle v,w\rangle\langle v^*,w^*\rangle_*.$$

The previous proposition implies that

$$f_k^{\sigma,\tau}(\cdot)=\langle\cdot\,\pi_n(\eta_k(q^\sigma)),\eta_1(q^\tau)\rangle_1=\langle\cdot\,\eta_k(q^\sigma),\eta_1(q^\tau)\rangle_1\in F^{\sigma,\tau}_{(1,k,0)}(0)$$

for $k=-1,0,1$ are ground states. Their radial parts $f_k^{\sigma,\tau}|_T\in\mathbb{C}[u^{\pm1}]$ can now be explicity evaluated as follows.

Lemma 5.14. *After suitable normalization, we have*

$$\begin{aligned}
f_{-1}^{\sigma,\tau}|_T&=\prod_{i=1}^n\big(1-q^{1-\sigma+\tau}u_i^{-1}\big)\big(1-q^{-3+\sigma-\tau}u_i\big),\\
f_{0}^{\sigma,\tau}|_T&=\prod_{i=1}^n\big(1-q^{1-\sigma+\tau}u_i^{-1}\big)\big(1+q^{-1-\sigma-\tau}u_i\big),\\
f_{1}^{\sigma,\tau}|_T&=\prod_{i=1}^n\big(1+q^{1+\sigma+\tau}u_i^{-1}\big)\big(1+q^{-3-\sigma-\tau}u_i\big)
\end{aligned}$$

in $\mathbb{C}[u^{\pm1}]$.

Proof. For $\lambda\in P_{2n}$, we write $|\lambda|_n=\lambda_1+\lambda_2+\cdots+\lambda_n$. By (5.15) we have

$$\begin{aligned}
K^{\lambda-\delta}\,\xi_{op}(s)&=(-1)^{\frac{n(n-1)}{2}}q^{|\lambda|_n-n^2}\xi(sq^{\lambda_1-\lambda_{2n}-1},sq^{\lambda_2-\lambda_{2n-1}-1},\ldots,sq^{\lambda_n-\lambda_{n+1}-1}),\\
K^{\lambda-\delta}\,\xi^*_{op}(s)&=(-1)^{\frac{n(n-1)}{2}}q^{-|\lambda|_n+n^2}\xi^*(sq^{1-\lambda_1+\lambda_{2n}},sq^{1-\lambda_2+\lambda_{2n-1}},\ldots,sq^{1-\lambda_n+\lambda_{n+1}})
\end{aligned}$$

for all $\lambda\in P_{2n}$. By (5.15) and (5.16), we conclude that

$$\begin{aligned}
\langle K^{\lambda-\delta}\,\xi_{op}(s),\xi_{op}(t)\rangle&=q^{|\lambda|_n-\frac{n(n+1)}{2}}\prod_{i=1}^n\big(1+s^{-1}t^{-1}q^{1+\lambda_{2n+1-i}-\lambda_i}\big),\\
\langle K^{\lambda-\delta}\,\xi^*_{op}(s),\xi^*_{op}(t)\rangle_*&=q^{-|\lambda|_n+\frac{3n(n+1)}{2}}\prod_{i=1}^n\big(1+s^{-1}t^{-1}q^{-1+\lambda_i-\lambda_{2n+1-i}}\big)
\end{aligned}$$

for all $\lambda\in P_{2n}$. The lemma now easily follows from the definition of the η_k's. □

Lemma 5.15. *Let $f \in F^{\sigma,\tau}_{(\kappa_1,\kappa_2,0)}(0)$ be a ground state. After a suitable renormalization of f, we have*

$$f|_T = u^{\delta(0,\kappa_1)} \prod_{i=1}^n \bigl(-q^{1+\sigma+\tau}u_i^{-1};q^2\bigr)_{\kappa_1+\kappa_2} \bigl(q^{1-\sigma+\tau}u_i^{-1};q^2\bigr)_{\kappa_1-\kappa_2}$$

in $\mathbb{C}[u^{\pm 1}]$.

Proof. We consider the $U_q(\mathfrak{g})$-module $W = (\Lambda^n(V) \otimes \Lambda^n(V^*))^{\otimes \kappa_1}$, with $*$-unitary inner product

$$\langle v_1 \otimes \cdots \otimes v_{\kappa_1}, w_1 \otimes \cdots \otimes w_{\kappa_1} \rangle_{\kappa_1} = \prod_{i=1}^{\kappa_1} \langle v_i, w_i \rangle_1$$

for all $v_i, w_j \in \Lambda^n(V) \otimes \Lambda^n(V^*)$. Lemma 5.10 implies that the nonzero vector

$$\chi_\tau = \eta_1(q^\tau) \otimes \eta_1(q^{\tau+2}) \otimes \cdots \otimes \eta_1(q^{\tau+2(\kappa_1-1)}) \in W$$

spans a copy of the one-dimensional $\mathcal{A}_\tau$-module $V(\kappa_1)_\tau$ within W^τ. By (5.17) we have

$$W \simeq L_{\delta(0,\kappa_1)^\natural} \oplus \bigoplus_{\lambda \in P_{2n}^+ : \lambda \prec \delta(0,\kappa_1)^\natural} L_\lambda^{\oplus d_\lambda} \tag{5.18}$$

as $U_q(\mathfrak{g})$-modules for certain multiplicities $d_\lambda \in \mathbb{Z}_{\geq 0}$. By Proposition 1.15, $\mathbb{C}\{\chi_\tau\}$ lies in the component $L_{\delta(0,\kappa_1)^\natural}$ of the decomposition (5.18).

For the realization of $V(\kappa_2)_\sigma$ in W^σ, we need to consider the cases $0 \leq \kappa_2 \leq \kappa_1$ and $-\kappa_1 \leq \kappa_2 \leq 0$ seperately. For $0 \leq \kappa_2 \leq \kappa_1$, Lemma 5.10 implies that the nonzero vector

$$\chi_\sigma^+ = \eta_0(q^\sigma)^{\otimes(\kappa_1-\kappa_2)} \otimes \eta_1(q^\sigma) \otimes \eta_1(q^{\sigma+2}) \otimes \cdots \otimes \eta_1(q^{\sigma+2(\kappa_2-1)}) \in W$$

spans a copy of the one-dimensional $\mathcal{A}_\sigma$-module $V(\kappa_2)_\sigma$ in W^σ. If $p : W \to L_{\delta(0,\kappa_1)^\natural}$ denotes the projection on $L_{\delta(0,\kappa_1)^\natural}$ along the decomposition (5.18), then $p(\mathbb{C}\{\chi_\sigma^+\})$ is a copy of the one-dimensional $\mathcal{A}_\sigma$-module $V(\kappa_2)_\sigma$ in $L^\sigma_{\delta(0,\kappa_1)^\natural}$ by highest weight considerations, cf. Lemma 2.5. Thus

$$f(\cdot) = \langle \cdot\, p(\chi_\sigma^+), \chi_\tau \rangle_{\kappa_1} = \langle \cdot\, \chi_\sigma^+, \chi_\tau \rangle_{\kappa_1}$$

defines a ground state $f \in F^{\sigma,\tau}_{(\kappa,\kappa_2,0)}(0)$. Since

$$f|_T = \prod_{i=1}^{\kappa_1-\kappa_2} f_0^{\sigma,\tau+2(i-1)}|_T \prod_{j=1}^{\kappa_2} f_1^{\sigma+2(j-1),\tau+2(\kappa_1-\kappa_2+j-1)}|_T,$$

the result now follows from the previous lemma by a straightforward manipulation of q-shifted factorials. For $-\kappa_1 \leq \kappa_2 \leq 0$ the vector χ_σ^+ should be replaced by

$$\chi_\sigma^- = \eta_0(q^\sigma)^{\otimes(\kappa_1+\kappa_2)} \otimes \eta_{-1}(q^\sigma) \otimes \eta_{-1}(q^{\sigma-2}) \otimes \cdots \otimes \eta_{-1}(q^{\sigma-2(-\kappa_2-1)}) \in W.$$

We leave the remaining straightforward computations to the reader. □

Theorem 1.21**(ii)** follows now immediately by combining Lemma 5.1, Corollary 5.9 and Lemma 5.15.

6. Generalized quantum Schur orthogonality relations

We freely use the notations of the previous sections. The normalized Haar functional $h : \mathbb{C}_q[G] \to \mathbb{C}$ is the unique linear map which intertwines the (left) regular $U_q(\mathfrak{g})$-action on $\mathbb{C}_q[G]$ with the trivial $U_q(\mathfrak{g})$-action on $\mathbb{C}$ and which maps $1 \in \mathbb{C}_q[G]$ to $1 \in \mathbb{C}$. In particular, h is identically zero on the summands $W(\lambda)$ ($\lambda \in P_{2n}^+ \setminus \{0\}$) in the Peter-Weyl decomposition (1.18).

The Haar functional h defines a pre-Hilbert space structure on $\mathbb{C}_q[G]$ by

$$\langle a, b\rangle_h = h(b^*a), \qquad \forall\, a, b \in \mathbb{C}_q[G].$$

The quantum Schur orthogonality relations imply that the Peter-Weyl decomposition (1.18) is an orthogonal decomposition with respect to $\langle\cdot,\cdot\rangle_h$.

We fix a scalar product $\langle\cdot,\cdot\rangle_{\vec{\kappa}}$ on the simple $U_q(\mathfrak{k})$-module $V(\kappa,\kappa_1)$ such that the representation map $\rho(\kappa,\kappa_1)$ is $*$-unitary (of course, $\langle\cdot,\cdot\rangle_{\vec{\kappa}}$ only depends on the parameters κ,κ_1 of $\vec{\kappa}$, and is unique up to scalar multiples). From Corollary 1.11 it follows that the simple $\mathcal{A}_\tau$-module $(\rho(\kappa,\kappa_1)_\tau, V(\kappa,\kappa_1)_\tau)$ is $*$-unitary with respect to $\langle\cdot,\cdot\rangle_{\vec{\kappa}}$.

Definition 6.1. *Let $\pi = \pi_{\vec{\kappa}}^{\sigma,\tau} : F_{\vec{\kappa}}^{\sigma,\tau} \times F_{\vec{\kappa}}^{\sigma,\tau} \to Hom(U_q(\mathfrak{g}),\mathbb{C})$ be the sesquilinear form*

$$\pi(f,g)(X) = \sum \langle f(K^{-\delta}X_1), g(K^{-\delta}\omega(X_2))\rangle_{\vec{\kappa}}, \qquad f,g \in F_{\vec{\kappa}}^{\sigma,\tau},\;\; X \in U_q(\mathfrak{g}).$$

Let $f,g \in F_{\vec{\kappa}}^{\sigma,\tau}$. We write $f\cdot K^{-\delta}$ for the right regular action of $K^{-\delta} \in U_q(\mathfrak{g})$ on the first tensor component of $f \in \mathbb{C}_q[G] \otimes V(\kappa,\kappa_1)_\tau$, and similarly for $g\cdot K^{-\delta}$. If

$$f\cdot K^{-\delta} = \sum f_i \otimes v_i, \qquad g\cdot K^{-\delta} = \sum g_j \otimes w_j,$$

with $f_i, g_j \in \mathbb{C}_q[G]$ and $v_i, w_j \in V(\kappa,\kappa_1)_\tau$, then the definition of the pairing π and (3.2) imply that

$$\pi(f,g)(X) = \sum f_i(X_1)g_j^*(X_2)\langle v_i, w_j\rangle_{\vec{\kappa}} = \sum \big(f_ig_j^*\big)(X)\langle v_i, w_j\rangle_{\vec{\kappa}} \tag{6.1}$$

for all $X \in U_q(\mathfrak{g})$. In particular, the image of π is contained in $\mathbb{C}_q[G]$. This observation can be strengthened as follows.

Proposition 6.2. *The image of the pairing $\pi_{\vec{\kappa}}^{\sigma,\tau}$ is contained in $\mathcal{H}^{\sigma,\tau}$.*

Proof. We use the characterization of the space $\mathcal{H}^{\sigma,\tau}$ as given in Remark 3.9**(ii)**.

Fix $f,g \in F_{\vec{\kappa}}^{\sigma,\tau}$. Let $a \in \mathcal{A}_\sigma$ and write $\Delta(a) = \sum a_1 \otimes a_2$ with the a_1's from $\mathcal{A}_\sigma$ (which can be done in view of Proposition 4.2). Using that the character $\chi(\kappa_2)_\sigma : \mathcal{A}_\sigma \to \mathbb{C}$ of $V(\kappa_2)_\sigma$ is $*$-unitary, $\chi(\kappa_2)_\sigma(a^*) = \overline{\chi(\kappa_2)_\sigma(a)}$ for $a \in \mathcal{A}_\sigma$, we obtain for $X \in U_q(\mathfrak{g})$,

$$\begin{aligned}\pi(f,g)(Xa) &= \sum \chi(\kappa_2)_\sigma(a_1)\,\langle f(K^{-\delta}X_1), g(K^{-\delta}\omega(X_2)\omega(a_2))\rangle_{\vec{\kappa}}\\ &= \sum \langle f(K^{-\delta}X_1), g(K^{-\delta}\omega(X_2)\omega(a_2)a_1^*)\rangle_{\vec{\kappa}}\\ &= \epsilon(a)\pi(f,g)(X)\end{aligned}$$

since

$$\sum \omega(a_2)a_1^* = \Big(\sum a_1S(a_2)\Big)^* = \overline{\epsilon(a)}1$$

by the antipode axiom for the Hopf algebra $U_q(\mathfrak{g})$. For the left invariance with respect to $\mathcal{A}_\tau$, let $b \in \mathcal{A}_\tau$, and write again $\Delta(b) = \sum b_1 \otimes b_2$ with all b_1's from $\mathcal{A}_\tau$. Now Δ is a $*$-algebra homomorphism and S is an anti-coalgebra homomorphism, so

$$\Delta(\omega_\delta(b)) = \sum \omega_\delta(b_2) \otimes \omega_\delta(b_1).$$

Furthermore, note that $(\omega \circ \omega_\delta)(X) = K^\delta X K^{-\delta}$ for $X \in U_q(\mathfrak{g})$. Combined with (3.6), we thus obtain for $X \in U_q(\mathfrak{g})$,

$$\begin{aligned}
\pi(f,g)(\omega_\delta(b)X) &= \sum \langle f(K^{-\delta}\omega_\delta(b_2)X_1), g(b_1K^{-\delta}\omega(X_2))\rangle_{\vec{\kappa}}\\
&= \sum \langle f(K^{-2\delta}\omega(b_2)K^\delta X_1), \rho(\kappa,\kappa_1)_\tau(b_1)g(K^{-\delta}\omega(X_2))\rangle_{\vec{\kappa}}\\
&= \sum \langle f(b_1^*S^2(\omega(b_2))K^{-\delta}X_1), g(K^{-\delta}\omega(X_2))\rangle_{\vec{\kappa}}\\
&= \overline{\epsilon(b)}\pi(f,g)(X),
\end{aligned}$$

since

$$\sum b_1^*S^2(\omega(b_2)) = \sum b_1^*S(b_2^*) = \epsilon(b^*)1 = \overline{\epsilon(b)}1.$$

$\square$

Choose a vector $v \in \widetilde{V}(\kappa,\kappa_1)_\tau$ such that $\langle v,v\rangle_{\vec{\kappa}} = 1$. Without loss of generality we may and will assume that the implicit identification $\widetilde{V}(\kappa,\kappa_1)_\tau \simeq \mathbb{C}$ in the definition of the restriction map $|_T$ is realized by the explicit map $zv \mapsto z$ ($z \in \mathbb{C}$).

Corollary 6.3. *If $f,g \in F_{\vec{\kappa}}^{\sigma,\tau}$, then $f|_T(g|_T)^* \in \mathbb{C}[u^{\pm1}]^W$.*

Proof. Let $f,g \in F_{\vec{\kappa}}^{\sigma,\tau}$. Since the Cartan elements K^λ ($\lambda \in P_{2n}$) are group-like and $*$-selfadjoint elements in $U_q(\mathfrak{g})$, it follows from Lemma 1.20 and from the definition of π that

$$\begin{aligned}
\mathrm{Res}_T(\pi(f,g))(q^\lambda) &= \pi(f,g)(K^\lambda)\\
&= \langle f(K^{\lambda-\delta}), g(K^{-\lambda-\delta})\rangle_{\vec{\kappa}}\\
&= f|_T(q^\lambda)(g|_T)^*(q^\lambda)
\end{aligned}$$

for $\lambda \in P_{2n}$, hence

$$\mathrm{Res}_T(\pi(f,g)) = f|_T(g|_T)^*$$

in $\mathbb{C}[u^{\pm1}]$. On the other hand, $\mathrm{Res}_T(\pi(f,g)) \in \mathbb{C}[u^{\pm1}]^W$ by the previous proposition and by Proposition 3.10. $\square$

We define a linear functional $h^{\sigma,\tau} : \mathbb{C}[u^{\pm1}] \to \mathbb{C}$ by integrating against the orthogonality measure of the Macdonald-Koornwinder polynomials $P_\mu^{\sigma,\tau}$ ($\mu \in \Lambda_n^+$), see (3.8). We renormalize $h^{\sigma,\tau}$ so that $h^{\sigma,\tau}(1) = 1$. In particular, for real parameters σ,τ satisfying

$$q^{\epsilon\sigma+\epsilon'\tau} < q^{-1}, \qquad \forall\, \epsilon,\epsilon' \in \{\pm1\}, \tag{6.2}$$

the functional $h^{\sigma,\tau}$ is given by

$$h^{\sigma,\tau}(p) = \frac{1}{N}\int_T p(u)\Delta_{\sigma,\tau}(u)\frac{du}{u}, \qquad p \in \mathbb{C}[u^{\pm1}],$$

with $\Delta_{\sigma,\tau}(u) = \Delta(u; -q^{\sigma+\tau+1}, -q^{-\sigma-\tau+1}, q^{\sigma-\tau+1}, q^{-\sigma+\tau+1}; q^2, q^2)$ the Macdonald-Koornwinder weight function and with positive normalization constant $N = \int_T \Delta_{\sigma,\tau}(u)du$. Then $h^{\sigma,\tau}$ gives rise to a pre-Hilbert structure on $\mathbb{C}[u^{\pm 1}]$,

$$\langle p, r\rangle_{\sigma,\tau} = h^{\sigma,\tau}(r^* p), \qquad p, r \in \mathbb{C}[u^{\pm 1}]$$

with $*$-structure on $\mathbb{C}[u^{\pm 1}]$ defined by (3.7). Theorem 1.21 for $\vec{\kappa} = \vec{0}$ and the orthogonality relations for the Macdonald-Koornwinder polynomials $P_\mu^{\sigma,\tau}$ imply $h(f) = h^{\sigma,\tau}(f|_T)$ for $f \in F_{\vec{0}}^{\sigma,\tau}$. By Proposition 3.8 this implies

$$h(f) = h^{\sigma,\tau}\big(\mathrm{Res}_T(f)\big), \qquad \forall\, f \in \mathcal{H}^{\sigma,\tau}. \tag{6.3}$$

Since $\mathrm{Res}_T : \mathcal{H}^{\sigma,\tau} \to \mathbb{C}[u^{\pm 1}]^W$ is a $*$-algebra isomorphism, we conclude from (6.3) that

$$\mathrm{Res}_T : (\mathcal{H}^{\sigma,\tau}, \langle\cdot,\cdot\rangle_h) \to (\mathbb{C}[u^{\pm 1}]^W, \langle\cdot,\cdot\rangle_{\sigma,\tau})$$

is an isomorphism of pre-Hilbert spaces.

Proposition 6.4. (Generalized quantum Schur orthogonality relations). *Let $\mu, \nu \in \Lambda_n^+$ and choose elementary vector valued spherical functions $f \in F_{\vec{\kappa}}^{\sigma,\tau}(\mu)$ and $g \in F_{\vec{\kappa}}^{\sigma,\tau}(\nu)$. If $\mu \neq \nu$, then $\langle f|_T, g|_T\rangle_{\sigma,\tau} = 0$.*

Proof. Let $f, g \in F_{\vec{\kappa}}^{\sigma,\tau}$. By the proof of Corollary 6.3, by Proposition 6.2 and by (6.3) we have

$$h\big(\pi(f,g)\big) = \langle f|_T, g|_T\rangle_{\sigma,\tau}. \tag{6.4}$$

Let $f \in F_{\vec{\kappa}}^{\sigma,\tau}(\mu)$ and $g \in F_{\vec{\kappa}}^{\sigma,\tau}(\nu)$ with $\mu, \nu \in \Lambda_n^+$ and $\mu \neq \nu$. We show that the left hand side of (6.4) vanishes. Since $f \cdot K^{-\delta} \in W(\mu^\natural) \otimes V(\kappa, \kappa_1)_\tau$ and $g \cdot K^{-\delta} \in W(\nu^\natural) \otimes V(\kappa, \kappa_1)_\tau$, we may write

$$f \cdot K^{-\delta} = \sum f_i \otimes v_i, \qquad g \cdot K^{-\delta} = \sum g_j \otimes w_j,$$

with $f_i \in W(\mu^\natural)$, $g_j \in W(\nu^\natural)$ and $v_i, w_j \in V(\kappa, \kappa_1)_\tau$. By (6.1) and by the quantum Schur orthogonality relations, we conclude that

$$h\big(\pi(f,g)\big) = \sum h(f_i g_j^*)\langle v_i, w_j\rangle_{\vec{\kappa}} = 0,$$

as desired. $\square$

The main application of the generalized quantum Schur orthogonality relations is the identification of the vector valued spherical functions with Macdonald-Koornwinder polynomials, as stated in Theorem 1.21**(iii)**.

Corollary 6.5. *Let $\mu \in \Lambda_n^+$ and choose elementary vector valued spherical functions $f_\mu \in F_{\vec{\kappa}}^{\sigma,\tau}(\mu)$ and $f_0 \in F_{\vec{\kappa}}^{\sigma,\tau}(0)$. Then*

$$\frac{f_\mu|_T}{f_0|_T} = D\, P_\mu\big(u; -q^{\sigma+\tau+1+\kappa_1+\kappa_2}, -q^{-\sigma-\tau+1}, q^{\sigma-\tau+1}, q^{-\sigma+\tau+1+\kappa_1-\kappa_2}; q^2, q^{2\kappa+2}\big)$$

for some nonzero constant D.

Proof. It suffices to prove the corollary for real parameters σ, τ satisfying (6.2). By Proposition 4.5 we can normalize the elementary vector valued spherical functions $f_\nu \in F_{\vec{\kappa}}^{\sigma,\tau}(\mu)$ ($\nu \in \Lambda_n^+$) such that

$$p_\nu := \frac{f_\nu|_T}{f_0|_T} = m_\nu + \sum_{\substack{\nu' \in \Lambda_n^+ \\ \nu' < \nu}} c_{\nu'} m_{\nu'} \in \mathbb{C}[u^{\pm 1}]^W \tag{6.5}$$

for some constants $c_{\nu'} \in \mathbb{C}$.

We fix now arbitrary $\mu \in \Lambda_n^+$ and we prove that p_μ is the monic Macdonald-Koornwinder polynomial of degree μ with suitable specialization of the parameters. Formula (6.5) applied to p_μ shows that

$$p_\mu = m_\mu + \sum_{\substack{\mu' \in \Lambda_n^+ \\ \mu' < \mu}} c_{\mu'} m_{\mu'} \in \mathbb{C}[u^{\pm 1}]^W \tag{6.6}$$

for some constants $c_{\mu'} \in \mathbb{C}$. By the generalized quantum Schur orthogonality relations we furthermore have

$$\int_T p_\mu(u) p_\nu^*(u) W(u) \frac{du}{u} = 0, \qquad \forall\, \nu \in \Lambda_n^+ :\, \nu \neq \mu \tag{6.7}$$

with weight function $W(u) = f_0|_T(u)(f_0|_T)^*(u)\Delta_{\sigma,\tau}(u)$. By Theorem 1.21**(ii)**, the weight function $W(u)$ ($u \in T$) can be expressed in terms of the Macdonald-Koornwinder weight function $\Delta(u; a, b, c, d; q, t)$ as

$$W(u) = C\, \Delta\bigl(u; -q^{\sigma+\tau+1+\kappa_1+\kappa_2}, -q^{-\sigma-\tau+1}, q^{\sigma-\tau+1}, q^{-\sigma+\tau+1+\kappa_1-\kappa_2}; q^2, q^{2\kappa+2}\bigr)$$

for some nonzero constant C. Properties (6.5) and (6.7) thus imply

$$\int_T p_\mu(u) m_\nu(u) W(u) \frac{du}{u} = 0 \qquad \forall\, \nu \in \Lambda_n^+ :\, \nu < \mu. \tag{6.8}$$

The proof now follows from the fact that the Macdonald-Koornwinder polynomial

$$P_\mu\bigl(u; -q^{\sigma+\tau+1+\kappa_1+\kappa_2}, -q^{-\sigma-\tau+1}, q^{\sigma-\tau+1}, q^{-\sigma+\tau+1+\kappa_1-\kappa_2}; q^2, q^{2\kappa+2}\bigr) \in \mathbb{C}[u^{\pm 1}]^W$$

has been defined as the unique W-invariant Laurent polynomial satisfying the properties (6.6) and (6.8). $\square$

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